<u>Theorem:</u> Let $B = \{\bar{w}_1, \ldots, \bar{w}_k\}$ be an <u>orthogonal</u> set of vectors in V and let $W = sp(B)$. For any vector $\bar{v}$ in V, we have

(i) $proj(\bar{v}, W)$ is the closest vector to $\bar{v}$ in W; that is, if $\bar{w} \in W$ then $\|\bar{v} - proj(\bar{v}, W)\| \le \|\bar{v} - \bar{w}\|$,

(ii) $proj(\bar{v}, W) = \bar{v}$ if and only if $\bar{v} \in W$,

(iii) $\bar{v}' = \bar{v} - proj(\bar{v}, W)$ is orthogonal to <u>every</u> vector in W, in particular, $\bar{v}'$ is orthogonal to $\bar{w}_1, \ldots, \bar{w}_k$.

<u>Proof</u>: See text, Problems 7.25, 7.30. □

This theorem underlies the <u>Gram-Schmidt Orthogonalization</u>

<u>Theorem</u>: Let $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be a basis for V and construct:

$$\bar{w}_1 = \bar{v}_1$$

$$\bar{w}_2 = \bar{v}_2 - \frac{\langle \bar{v}_2, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle}\bar{w}_1 = \bar{v}_2 - proj(\bar{v}_2, sp\{\bar{w}_1\})$$

$$\bar{w}_3 = \bar{v}_3 - \frac{\langle \bar{v}_3, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle}\bar{w}_1 - \frac{\langle \bar{v}_3, \bar{w}_2 \rangle}{\langle \bar{w}_2, \bar{w}_2 \rangle}\bar{w}_2 = \bar{v}_3 - proj(\bar{v}_3, sp\{\bar{w}_1, \bar{w}_2\})$$

$$\vdots$$

$$\bar{w}_n = \bar{v}_n - \frac{\langle \bar{v}_n, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle}\bar{w}_1 - \cdots - \frac{\langle \bar{v}_n, \bar{w}_{n-1} \rangle}{\langle \bar{w}_{n-1}, \bar{w}_{n-1} \rangle}\bar{w}_{n-1}$$

$$= \bar{v}_n - proj(\bar{v}_n, sp\{\bar{w}_1, \ldots, \bar{w}_{n-1}\}).$$

Then $B' = \{\bar{w}_1, \ldots, \bar{w}_n\}$ is an <u>orthogonal</u> basis for V.

□

[Observe that if we define $W_i = sp\{\bar{w}_1, \ldots, \bar{w}_i\}$ for each i, then $\bar{w}_{i+1} = \bar{v}_{i+1} - proj(\bar{v}_{i+1}, W_i)$, thus $\bar{w}_{i+1}$ is the component of $\bar{v}_{i+1}$ orthogonal to the subspace W_i.]