Corollary: If V is a finite dimensional inner product space, then V has an orthogonal basis. Moreover, if W is a subspace of V, then W has an orthogonal basis, and any orthogonal basis for W can be extended to an orthogonal basis for V.

Remarks ① if $\{\bar{v}_1, \cdots, \bar{v}_n\} = B$ is orthogonal, then $B' = B$.
② the Gram-Schmidt process fails if the original set is linearly dependent.
③ At each step in the process, we can replace $\bar{w}_j$ with $\alpha\bar{w}_j$ for any scalar $\alpha \neq 0$ and still get an orthogonal basis. (So we can "clear fractions" at each stage, for example.)

EXAMPLE: Let $V = sp\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \subseteq \mathbb{R}^4$, where
$\bar{v}_1 = (1,2,1,2)$, $\bar{v}_2 = (2,1,3,1)$, $\bar{v}_3 = (3,2,1,1)$ [Check: independent.]
We use the Gram-Schmidt process to find an orthogonal basis for V.

$$\boxed{\bar{w}_1 = \bar{v}_1 = (1,2,1,2)}$$

$$\bar{w}_2 = \bar{v}_2 - \frac{\langle \bar{v}_2, \bar{w}_1\rangle}{\langle \bar{w}_1, \bar{w}_1\rangle}\bar{w}_1 = (2,1,3,1) - \frac{2+2+3+2}{1+4+1+4}(1,2,1,2) = (2,1,3,1) - \frac{9}{10}(1,2,1,2)$$
$$= (2,1,3,1) - \left(\tfrac{9}{10}, \tfrac{18}{10}, \tfrac{9}{10}, \tfrac{18}{10}\right) = \left(\tfrac{11}{10}, -\tfrac{8}{10}, \tfrac{21}{10}, -\tfrac{8}{10}\right).$$

We replace $\bar{w}_2$ with $10\bar{w}_2$ and let $\boxed{\bar{w}_2 = (11,-8,21,-8)}$

$$\bar{w}_3 = \bar{v}_3 - \frac{\langle \bar{v}_3, \bar{w}_1\rangle}{\langle \bar{w}_1, \bar{w}_1\rangle}\bar{w}_1 - \frac{\langle \bar{v}_3, \bar{w}_2\rangle}{\langle \bar{w}_2, \bar{w}_2\rangle}\bar{w}_2 =$$
$$= (3,2,1,1) - \frac{3+4+1+2}{1+4+1+4}(1,2,1,2) - \frac{33-16+21-8}{121+64+441+64}(11,-8,21,-8)$$
$$= (3,2,1,1) - \frac{10}{10}(1,2,1,2) - \frac{30}{690}(11,-8,21,-8)$$
$$= (3,2,1,1) - (1,2,1,2) - \frac{1}{23}(11,-8,21,-8) = (2,0,0,-1) + \left(-\tfrac{11}{23}, \tfrac{8}{23}, -\tfrac{21}{23}, \tfrac{8}{23}\right)$$
$$= \left(\tfrac{35}{23}, \tfrac{8}{23}, -\tfrac{21}{23}, -\tfrac{15}{23}\right).$$

We replace $\bar{w}_3$ with $23\bar{w}_3$ to obtain $\boxed{\bar{w}_3 = (35,8,-21,-15)}$.

check that $\boxed{B' = \{(1,2,1,2), (11,-8,21,-8), (35,8,-21,-15)\}}$ is orthogonal.

$\boxed{\text{END LEC 16}}$