(48) Lec 17, 3/4/09

Linear Transformations [§5.3]

*[Read §§5.1, 5.2 carefully — this will be assumed.]

Defn: Let V, W be vector spaces over a field K. A map $\varphi: V \to W$ is a <u>linear map</u> or <u>linear transformation</u> if
(i) $\varphi(\bar{u}+\bar{v}) = \varphi(\bar{u}) + \varphi(\bar{v})$ for all $\bar{u}, \bar{v} \in V$, and
(ii) $\varphi(\alpha \bar{v}) = \alpha \varphi(\bar{v})$ for all $\alpha \in K$, $\bar{v} \in V$.

Proposition: If $\varphi: V \to W$ is a linear transformation, then $\varphi(\bar{0}_V) = \bar{0}_W$.

Proof: Since $\bar{0}_V = 0 \cdot \bar{0}_V$ and φ is linear, we have
$$\varphi(\bar{0}_V) = \varphi(0\bar{0}_V) = 0\varphi(\bar{0}_V) = \bar{0}_W. \quad \square$$

The proposition also shows that any map φ that does <u>not</u> send $\bar{0}_V$ to $\bar{0}_W$ <u>cannot</u> be linear.

Examples:

① The zero mapping ($\bar{v} \mapsto \bar{0}_W$ all $\bar{v} \in V$) and identity mapping ($\bar{v} \mapsto \bar{v}$ all $\bar{v} \in V$) are linear. (See text.)

② $\varphi: \mathbb{R}^3 \to \mathbb{R}^2$ given by $\varphi((x,y,z)) = (x+y, y+z)$ is linear.
Proof. Let $\bar{v} = (x,y,z)$, $\bar{w} = (x', y', z')$. Then
$$\varphi(\bar{v}+\bar{w}) = \varphi((x+x', y+y', z+z')) = ((x+x')+(y+y'), (y+y')+(z+z'))$$
$$= ((x+y)+(x'+y'), (y+z)+(y'+z')) =$$
$$= (x+y, y+z) + (x'+y', y'+z') = \varphi(\bar{v}) + \varphi(\bar{w}),$$
and $\varphi(\alpha\bar{v}) = \varphi((\alpha x, \alpha y, \alpha z)) = (\alpha x + \alpha y, \alpha y + \alpha z)$
$$= (\alpha(x+y), \alpha(y+z)) = \alpha(x+y, y+z) = \alpha\varphi(\bar{v}). \quad \square$$

③ Let V be a vector space over K with basis $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$. The map $\varphi: V \to K^n$ defined by $\varphi(\bar{v}) = [\bar{v}]_B$ is linear:
$$\varphi(\bar{v}+\bar{w}) = [\bar{v}+\bar{w}]_B = [\bar{v}]_B + [\bar{w}]_B = \varphi(\bar{v}) + \varphi(\bar{w}),$$
$$\varphi(\alpha\bar{v}) = [\alpha\bar{v}]_B = \alpha[\bar{v}]_B = \alpha\varphi(\bar{v}). \quad \square$$