(EXAMPLES, cont.)

(4) Let A be a fixed $m\times n$ matrix. Define $\varphi_A: \mathbb{R}^n \to \mathbb{R}^m$ by $\varphi_A(\bar{v}) = A\bar{v}$. Then φ_A is linear by properties of matrix multiplication:

$$\varphi_A(\bar{v}+\bar{w}) = A(\bar{v}+\bar{w}) = A\bar{v}+A\bar{w} = \varphi(\bar{v})+\varphi(\bar{w})$$
$$\varphi(\alpha\bar{v}) = A(\alpha\bar{v}) = \alpha(A\bar{v}) = \alpha\varphi(\bar{v}). \quad \square$$

Using induction, we can extend the linearity conditions to any linear combination:

Proposition: If $\varphi: V\to W$ is a linear transformation, then for all $\alpha_i\in K$, $\bar{v}_i\in V$,
$$\varphi(\alpha_1\bar{v}_1+\cdots+\alpha_k\bar{v}_k) = \alpha_1\varphi(\bar{v}_1)+\cdots+\alpha_k\varphi(\bar{v}_k).$$

This result implies that a linear transformation is completely determined by its action on a basis. If $B=\{\bar{v}_1,\dots,\bar{v}_n\}$ is a basis for V and $\bar{v}\in V$, then $\bar{v} = \alpha_1\bar{v}_1+\cdots+\alpha_n\bar{v}_n$, $\alpha_i\in K$, and $\varphi(\bar{v}) = \alpha_1\varphi(\bar{v}_1)+\cdots+\alpha_n\varphi(\bar{v}_n)$.

Example: Suppose $\varphi: \mathbb{R}^2\to\mathbb{R}^2$ is linear and $\varphi((1,0)) = (2,3)$, $\varphi((0,1)) = (4,5)$. Then
$$\varphi((7,8)) = \varphi(7(1,0)+8(0,1)) = 7\varphi((1,0))+8\varphi((0,1))$$
$$= 7(2,3)+8(4,5) = (46,61).$$
In general, $\varphi((a,b)) = a\varphi((1,0))+b\varphi((0,1)) = a(2,3)+b(4,5) = (2a+4b, 3a+5b)$. $\square$

Conversely, we can define a linear map by sending basis vectors to any vectors we wish:

Theorem: Let V, W be vector spaces over K and let $B=\{\bar{v}_1,\dots,\bar{v}_n\}$ be a basis for V. Let $\bar{w}_1,\dots,\bar{w}_n$ be ANY vectors in W. There is a unique linear map $\varphi: V\to W$ with $\varphi(\bar{v}_i) = \bar{w}_i$ for all i, given by
$$\varphi(\bar{v}) = \alpha_1\varphi(\bar{v}_1)+\cdots+\alpha_n\varphi(\bar{v}_n) = \alpha_1\bar{w}_1+\cdots+\alpha_n\bar{w}_n$$
for $\bar{v} = \alpha_1\bar{v}_1+\cdots+\alpha_n\bar{v}_n$ in V. [See Problem 5.13.]