Basic Properties of Vector Spaces

Theorem: Let V be a vectorspace over a field K.
(i) The zero vector of V is unique
(ii) For each $\bar{v} \in V$, the additive inverse of $\bar{v}$ is unique.

Proof: (i) Suppose V has two vectors $\bar{0}_1, \bar{0}_2$ that satisfy the zero vector axiom [A2]
Then $\bar{0}_1 + \bar{0}_2 = \bar{0}_2$ because $\bar{0}_1$ satisfies [A2] and
$\bar{0}_1 + \bar{0}_2 = \bar{0}_1$ because $\bar{0}_2$ satisfies [A2].
Hence $\bar{0}_1 = \bar{0}_1 + \bar{0}_2 = \bar{0}_2$ and so $\bar{0}_1 = \bar{0}_2$.

(ii) Let $\bar{v} \in V$ and suppose there are two vectors $\bar{u}_1, \bar{u}_2 \in V$ satisfying the additive inverse property [A3], that is,
$$\bar{u}_1 + \bar{v} = \bar{v} + u_1 = \bar{0} \text{ and } \bar{u}_2 + \bar{v} = \bar{v} + \bar{u}_2 = \bar{0}.$$
We then have
$$\begin{aligned}
\bar{u}_1 &= \bar{u}_1 + \bar{0} && \text{by [A2]} \\
&= \bar{u}_1 + (\bar{v} + \bar{u}_2) && \text{by choice of } \bar{u}_2, \text{ [A3]} \\
&= (\bar{u}_1 + \bar{v}) + \bar{u}_2 && \text{by associativity, [A1]} \\
&= \bar{0} + \bar{u}_2 && \text{by choice of } \bar{u}_1, \text{ [A3]} \\
&= \bar{u}_2 && \text{by [A2].}
\end{aligned}$$
Hence $\bar{u}_1 = \bar{u}_2$. $\square$

Theorem (Cancellation) Let V be a vector space over K.
If $\bar{u}, \bar{v}, \bar{w}$ are vectors in V with $\bar{u} + \bar{w} = \bar{v} + \bar{w}$, then $u = v$.

Proof: HW1, problem #4.
[Hint: Add $-\bar{w}$ to both sides of $\bar{u} + \bar{w} = \bar{v} + \bar{w}$ and use axioms. Be sure to state which axiom is used for each step.]