Isomorphisms

Defn: A linear transformation $\varphi: V \to W$ is an isomorphism if φ is both one-to-one and onto. In this case, we say V is isomorphic to W and write $V \cong W$.

Isomorphic vector spaces are "algebraically" the same, except for notation of vectors.

EXAMPLE: If V is a vector space over K of dimension n and basis B, then $\varphi: V \to K^n$ given by $\varphi(\bar{v}) = [\bar{v}]_B$ is an isomorphism. Hence if $\dim V = n$ then $V \cong K^n$.

In particular, $P_n(t) \cong \mathbb{R}^{n+1}$ and $M_{m,n}(\mathbb{R}) \cong \mathbb{R}^{mn}$.

Recall that if $\varphi: V \to W$ is any map that is one to one and onto, there is an inverse map $\varphi^{-1}: W \to V$ defined by $\varphi^{-1}(\bar{w}) = \bar{v}$ if and only if $\varphi(\bar{v}) = \bar{w}$.
Moreover, if φ is linear, then φ^{-1} is also linear:

Proposition: If $\varphi: V \to W$ is a linear transformation that is one to one and onto, then the inverse map $\varphi^{-1}: W \to V$ is a linear transformation.

End Lec 17 Pf: [See Problem 5.15] $\square$

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Kernel and Image (§5.4)

Defn: Let $\varphi: V \to W$ be a linear transformation.
The Kernel of φ is $\ker \varphi = \{\bar{v} \in V \mid \varphi(\bar{v}) = \bar{0}_W\}$, the set of vectors of V mapped to $\bar{0}_W$ by φ.
The image of φ is $\operatorname{Im} \varphi = \{\bar{w} \in W \mid \bar{w} = \varphi(\bar{v}) \text{ some } \bar{v} \in V\}$, the set of vectors of W that are the image of some vector in V under φ.