Theorem If $\varphi: V \to W$ is a linear transformation, then
(i) $\ker\varphi$ is a subspace of V, and
(ii) $\operatorname{Im}\varphi$ is a subspace of W.

Proof: (i) We know $\varphi(\bar{0}_V) = \bar{0}_W$, so $\bar{0}_V \in \ker\varphi$ and $\ker\varphi$ is nonempty.
If $\bar{v}_1, \bar{v}_2 \in \ker\varphi$, then $\varphi(\bar{v}_1) = \varphi(\bar{v}_2) = \bar{0}_W$, and by linearity of φ,
$\varphi(\bar{v}_1 + \bar{v}_2) = \varphi(\bar{v}_1) + \varphi(\bar{v}_2) = \bar{0}_W + \bar{0}_W = \bar{0}_W$. Hence $\underline{\bar{v}_1 + \bar{v}_2 \in \ker\varphi}$.
If $\alpha \in K$ and $\bar{v} \in \ker\varphi$, then $\varphi(\bar{v}) = \bar{0}_W$, and by linearity of φ,
$\varphi(\alpha\bar{v}) = \alpha\varphi(\bar{v}) = \alpha\bar{0}_W = \bar{0}_W$, and so $\underline{\alpha\bar{v} \in \ker\varphi}$.

(ii) Again, $\bar{0}_V \in V$ and $\varphi(\bar{0}_V) = \bar{0}_W$, so $\bar{0}_W \in \operatorname{Im}\varphi$. Hence $\underline{\operatorname{Im}\varphi \neq \phi}$.
Let $\bar{w}_1, \bar{w}_2 \in \operatorname{Im}\varphi$. Then $\bar{w}_1 = \varphi(\bar{v}_1)$, $\bar{w}_2 = \varphi(\bar{v}_2)$ for some $\bar{v}_1, \bar{v}_2 \in V$.
By linearity of φ, $\bar{w}_1 + \bar{w}_2 = \varphi(\bar{v}_1) + \varphi(\bar{v}_2) = \varphi(\bar{v}_1 + \bar{v}_2)$, and $\bar{v}_1 + \bar{v}_2 \in V$,
hence $\underline{\bar{w}_1 + \bar{w}_2 \in \operatorname{Im}\varphi}$.
Let $\alpha \in K$, $\bar{w} \in \operatorname{Im}\varphi$, so that $\bar{w} = \varphi(\bar{v})$ for some $\bar{v} \in V$.
By linearity of φ, $\alpha\bar{w} = \alpha\varphi(\bar{v}) = \varphi(\alpha\bar{v})$, and $\alpha\bar{v} \in V$, hence
$\underline{\alpha\bar{w} \in \operatorname{Im}\varphi}$. □

More generally, let $\varphi: V \to W$ be a linear transformation.
If S is a subset of V, the <u>image of S under φ</u> is
$\varphi(S) = \{\bar{w} \in W \mid \bar{w} = \varphi(\bar{s}) \text{ some } \bar{s} \in S\} = \{\varphi(\bar{s}) \mid \bar{s} \in S\}$.

If T is a subset of W, the <u>preimage of T under φ</u> is
$\varphi^{-1}(T) = \{\bar{v} \in V \mid \varphi(\bar{v}) \in T\}$.

<u>NOTES</u>: ① $\varphi^{-1}(T)$ is simply notation for a <u>subset</u> of V; it does <u>not</u> indicate that φ has an inverse <u>map</u>.

② The kernel of φ is the preimage of $\bar{0}_W$ under φ, that is,
$$\ker\varphi = \varphi^{-1}(\{\bar{0}_W\}).$$

③ The image of φ is $\varphi(V)$, the image of V under φ.