The proof of the following generalization of the theorem above is a homework problem:

Theorem: Let $\varphi : V \to W$ be a linear transformation.
(i) If $U$ is a subspace of $W$, then $\varphi^{-1}(U)$ is a subspace of $V$.
(ii) If $U$ is a subspace of $V$, then $\varphi(U)$ is a subspace of $W$.

EXAMPLE: Let $A$ be an $m \times n$ matrix and let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $\varphi_A$; that is $\varphi(\bar{v}) = A\bar{v}$, $\bar{v} \in \mathbb{R}^n$. We have:

(1) $\ker \varphi = \{\bar{v} \in \mathbb{R}^n \mid \varphi(\bar{v}) = \bar{0}\} = \{\bar{v} \in \mathbb{R}^n \mid A\bar{v} = \bar{0}\} = \mathcal{N}(A)$, i.e., $\ker \varphi_A$ is the null space of $A$ and so $\dim \ker \varphi$ is the nullity of $A$.

(2) $\operatorname{Im} \varphi = \{\bar{w} \in \mathbb{R}^m \mid \bar{w} = \varphi(\bar{v})$ some $\bar{v} \in \mathbb{R}^n\}$
$= \{\bar{w} \in \mathbb{R}^m \mid \bar{w} = A\bar{v}$ some $\bar{v} \in \mathbb{R}^n\} = \mathcal{C}(A)$,
i.e., $\operatorname{Im} \varphi_A$ is the column space of $A$ and so $\dim \operatorname{Im} \varphi$ is the rank of $A$.

We make the following definition accordingly:

Defn: If $\varphi : V \to W$ is a linear transformation,
(i) the nullity of $\varphi$ is $\dim(\ker \varphi)$,
(ii) the rank of $\varphi$ is $\dim(\operatorname{Im} \varphi)$.

We generalize the Rank + Nullity Theorem (p 27 of notes):

Theorem (Rank + Nullity) If $V, W$ are finite dimensional vector spaces over $K$ and $\varphi : V \to W$ is a linear transformation, then

$$\operatorname{rank} \varphi + \operatorname{nullity} \varphi = \dim V,$$

that is,

$$\dim \operatorname{Im} \varphi + \dim \ker \varphi = \dim V.$$