Proof: Let $B_1 = \{\bar{u}_1, \cdots, \bar{u}_m\}$ be a basis for $\ker \varphi$, so $\dim \ker \varphi = m$.
Extend B_1 to a basis for V, say
$$B = \{\bar{u}_1, \cdots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_r\},$$
so $\dim V = m + r$.

We will show that $B_2 = \{\varphi(\bar{v}_1), \ldots, \varphi(\bar{v}_r)\}$ is a basis for $\mathrm{Im}\,\varphi$, hence $\dim \mathrm{Im}\,\varphi = r$ and
$$\dim V = m + r = \dim \ker \varphi + \dim \mathrm{Im}\,\varphi,$$
as claimed.

B_2 spans $\mathrm{Im}\,\varphi$:

Let $\bar{\omega} \in \mathrm{Im}\,\varphi$, so that $\bar{\omega} = \varphi(\bar{v})$ for some $\bar{v} \in V$.
Since B is a basis for V, we can write
$$\bar{v} = \alpha_1 \bar{u}_1 + \cdots + \alpha_m \bar{u}_m + \beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r$$
with $\alpha_i, \beta_j \in K$, and so since φ is linear,
$$\begin{aligned}\bar{\omega} = \varphi(\bar{v}) &= \varphi(\alpha_1 \bar{u}_1 + \cdots + \alpha_m \bar{u}_m + \beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r) \\ &= \alpha_1 \varphi(\bar{u}_1) + \cdots + \alpha_m \varphi(\bar{u}_m) + \beta_1 \varphi(\bar{v}_1) + \cdots + \beta_r \varphi(\bar{v}_r) \\ &= \beta_1 \varphi(\bar{v}_1) + \cdots + \beta_r \varphi(\bar{v}_r),\end{aligned}$$
as $\bar{u}_i \in \ker \varphi$ for all i ($\varphi(\bar{u}_i) = \bar{0}_W$). Hence $\bar{\omega}$ is a linear combination of B_2, and so B_2 spans $\mathrm{Im}\,\varphi$.

B_2 is linearly independent.

Suppose $\beta_1 \varphi(\bar{v}_1) + \cdots + \beta_r \varphi(\bar{v}_r) = \bar{0}_W$. Since φ is linear, this means $\bar{0}_W = \varphi(\beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r)$ and $\beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r \in \ker \varphi$.
Since B_1 is a basis for $\ker \varphi$, we have
$$\beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r = \gamma_1 \bar{u}_1 + \cdots + \gamma_m \bar{u}_m$$
for some $\gamma_i \in K$. But then
$$\beta_1 \bar{v}_1 + \cdots + \beta_r \bar{v}_r + (-\gamma_1)\bar{u}_1 + \cdots + (-\gamma_m)\bar{u}_m = \bar{0}_V,$$
and since B is a basis for V, hence linearly independent, this implies $\beta_1 = \cdots = \beta_r = \gamma_1 = \cdots = \gamma_m = 0$. In particular, we have shown that if $\beta_1 \varphi(\bar{v}_1) + \cdots + \beta_r \varphi(\bar{v}_r) = \bar{0}_W$, then $\beta_1 = \cdots = \beta_r = 0$, and so B_2 is linearly independent. $\square$

[End Lec 18] [See Problems .23 for a slightly different proof.]