(54) Lec 19, 3/9/09

Singular and Nonsingular Maps

Recall that a map $f: S \to T$ is one to one if $s_1 \neq s_2$ in S implies $f(s_1) \neq f(s_2)$, or equivalently, if $f(s_1) = f(s_2)$ implies $s_1 = s_2$.

If $\varphi: V \to W$ is any map between vector spaces (linear or not) such that φ is one to one and $\varphi(\bar{0}_V) = \bar{0}_W$, then $\bar{0}_V$ is the only vector in V mapped to $\bar{0}_W$.
If φ is linear, this "local one-to-one-ness" is enough to imply φ is one to one:

Theorem: Let $\varphi: V \to W$ be a linear transformation. Then φ is one-to-one if and only if $\ker \varphi = \{\bar{0}_V\}$

Proof: $\boxed{\Rightarrow}$ Since φ is linear, $\varphi(\bar{0}_V) = \bar{0}_W$, so $\bar{0}_V \in \ker \varphi$. If φ is one to one $\bar{0}_V$ is the only vector mapped to $\bar{0}_W$, so $\ker \varphi = \{\bar{0}_V\}$.

$\boxed{\Leftarrow}$ Let $\ker \varphi = \{\bar{0}_V\}$ and suppose $\bar{v}_1, \bar{v}_2 \in V$ with $\varphi(\bar{v}_1) = \varphi(\bar{v}_2)$. Then $\bar{0}_W = \varphi(\bar{v}_1) - \varphi(\bar{v}_2) = \varphi(\bar{v}_1 - \bar{v}_2)$, since φ is linear. Hence $\bar{v}_1 - \bar{v}_2 \in \ker \varphi = \{\bar{0}_V\}$, so $\bar{v}_1 - \bar{v}_2 = \bar{0}_V$ and $\bar{v}_1 = \bar{v}_2$. Therefore φ is one-to-one. □

Defn: A linear map $\varphi: V \to W$ is nonsingular if $\ker \varphi = \{\bar{0}_V\}$ (or equivalently, φ is one to one). Otherwise, we say φ is singular.