Linear Maps and Spanning/Independence/Bases

Theorem: Let $\varphi: V \to W$ be a linear transformation. If S spans V then $\varphi(S) = \{\varphi(\bar{s}) \mid \bar{s} \in S\}$ spans $\operatorname{Im}\varphi$.

Proof: By definition, $\varphi(S)$ is a subset of $\operatorname{Im}\varphi$. If $\bar{w} \in \operatorname{Im}\varphi$, then $\bar{w} = \varphi(\bar{v})$ for some $\bar{v} \in V$, and since S spans V, we have $\bar{v} = \alpha_1 \bar{s}_1 + \cdots + \alpha_n \bar{s}_n$ for some $\alpha_i \in k$, $\bar{s}_i \in S$. By linearity of φ, $\bar{w} = \varphi(\bar{v}) = \alpha_1 \varphi(\bar{s}_1) + \cdots + \alpha_n \varphi(\bar{s}_n)$, and each $\varphi(\bar{s}_i)$ is in $\varphi(S)$. Hence every vector $\bar{w}$ in $\operatorname{Im}\varphi$ is a linear combination of vectors in $\varphi(S)$, so $\varphi(S)$ spans $\operatorname{Im}\varphi$. $\square$

Note: Given a spanning set $\{\bar{v}_1, \dots, \bar{v}_n\}$ for V, we have that $\{\varphi(\bar{v}_1), \dots, \varphi(\bar{v}_n)\}$ is a spanning set for $\operatorname{Im}\varphi$. We can find a basis for $\operatorname{Im}\varphi$ by methods introduced previously.

Corollary: Let $\varphi: V \to W$ be a linear transformation and let S be a spanning set for V. The set $\varphi(S) = \{\varphi(\bar{s}) \mid \bar{s} \in S\}$ is a spanning set for W if and only if φ is onto.
[That is, φ sends spanning sets to spanning sets if and only if φ is onto.]

In general, if S is linearly independent, $\varphi(S)$ need not be independent. However independence of $\varphi(S)$ does imply the independence of S:

Theorem: Let $\varphi: V \to W$ be a linear transformation, S a subset of V. If $\varphi(S)$ is linearly independent, then S is linearly independent.

Proof: Suppose $\alpha_1 \bar{s}_1 + \cdots + \alpha_k \bar{s}_k = \bar{0}_V$, $\alpha_i \in k$, $\bar{s}_i \in S$. Since φ is linear, $\bar{0}_W = \varphi(\bar{0}_V) = \alpha_1 \varphi(\bar{s}_1) + \cdots + \alpha_k \varphi(\bar{s}_k)$, and each $\varphi(\bar{s}_i)$ is in $\varphi(S)$. Since $\varphi(S)$ is linearly independent, $\alpha_1 = \cdots = \alpha_k = 0$. Hence S is linearly independent. $\square$