For a given linear map $\varphi: V \to W$, there may be some independent sets S for which $\varphi(S)$ is independent and others for which $\varphi(S)$ is <u>not</u> independent.

<u>Theorem:</u> Let $\varphi: V \to W$ be a linear transformation. Then $\varphi(S)$ is linearly independent for <u>every</u> linearly independent subset S of V if and only if φ is one to one.

<u>Proof:</u> $\boxed{\Rightarrow}$ Suppose $\varphi(S)$ is independent whenever $S \subseteq V$ is. In particular, if $\bar{v} \in V$ is nonzero, then $\{\bar{v}\}$ is linearly independent, and so $\{\varphi(\bar{v})\}$ is independent. Thus $\varphi(\bar{v}) \neq \bar{0}_W$. Hence $\ker\varphi = \{\bar{0}_V\}$ and φ is one to one.

$\boxed{\Leftarrow}$ Suppose φ is one to one, so $\ker\varphi = \{\bar{0}_V\}$, and let $S \subseteq V$ be linearly independent.

Let $\alpha_1\bar{w}_1 + \cdots + \alpha_k\bar{w}_k = \bar{0}_W$ for $\alpha_i \in K$, $\bar{w}_i \in \varphi(S)$. Then $\bar{w}_i = \varphi(\bar{s}_i)$ some $\bar{s}_i \in S$ for each i, and so $\bar{0}_W = \alpha_1\varphi(\bar{s}_1) + \cdots + \alpha_k\varphi(\bar{s}_k) = \varphi(\alpha_1\bar{s}_1 + \cdots + \alpha_k\bar{s}_k)$, since φ is linear. Hence $\alpha_1\bar{s}_1 + \cdots + \alpha_k\bar{s}_k \in \ker\varphi = \{\bar{0}_V\}$, and so $\alpha_1\bar{s}_1 + \cdots + \alpha_k\bar{s}_k = \bar{0}_V$. By independence of S, we have $\alpha_1 = \cdots = \alpha_k = 0$, and so $\varphi(S)$ is linearly independent. $\square$

Combining this theorem and the corollary above yields:

<u>Theorem:</u> Let $\varphi: V \to W$ be a linear transformation. The map φ sends bases of V to bases of W if and only if φ is one to one and onto, i.e., an isomorphism.

End Lec 19