Linear Operators and Invertible Maps

Theorem Let V, W be finite dimensional vector spaces over K with $\dim V = \dim W$. If $\varphi : V \to W$ is a linear transformation, then φ is one to one if and only if φ is onto.

Proof: By the Rank + Nullity Theorem, we have

$$(*) \quad \dim \operatorname{Im}\varphi + \dim \ker\varphi = \dim V = \dim W.$$

$\boxed{\Rightarrow}$ If φ is one to one, then $\ker\varphi = \{\bar{0}_V\}$ and $\dim \ker\varphi = 0$. Hence $(*)$ becomes $\dim \operatorname{Im}\varphi = \dim W$. Since W is finite dimensional and $\operatorname{Im}\varphi$ is a subspace of W, this implies $\operatorname{Im}\varphi = W$ and φ is onto.

$\boxed{\Leftarrow}$ If φ is onto, then $\operatorname{Im}\varphi = W$, and so $\dim \operatorname{Im}\varphi = \dim W$. Hence $(*)$ implies $\dim \ker\varphi = 0$, and so $\ker\varphi = \{\bar{0}_V\}$. Therefore φ is one to one. $\square$

Defn: A linear transformation $\varphi : V \to V$ is called a linear operator on V.

Corollary: Let φ be a linear operator on a finite dimensional vector space V. The following are equivalent:

(i) $\ker\varphi = \{\bar{0}_V\}$.
(ii) φ is one to one.
(iii) φ is onto.
(iv) φ is invertible.

Remark: The corollary says that for a linear operator on a finite dimensional vector space, "invertible" is equivalent to "nonsingular."

In particular, if A is an $n \times n$ matrix over $\mathbb{R}$, then $\varphi_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $\varphi_A(\bar{x}) = A\bar{x}$ is nonsingular if and only if A is an invertible matrix. [Exercise.]