Matrices and Linear Maps (§6.2, §6.5)

Let V be a vector space over K with basis $\{\bar{v}_1, \dots, \bar{v}_n\} = S$ and W a vector space over K with basis $\{\bar{w}_1, \dots, \bar{w}_m\} = S'$.
We can coordinatize V and W relative to these bases:

If $\bar{v} = \alpha_1 \bar{v}_1 + \dots + \alpha_n \bar{v}_n \in V$, then $[v]_S = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ and

if $\bar{w} = \beta_1 \bar{w}_1 + \dots + \beta_m \bar{w}_m \in W$, then $[\bar{w}]_{S'} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$.

[Note: we now must use column vectors for $[\bar{v}]_S, [\bar{w}]_{S'}$.]

Suppose $\varphi: V \to W$ is a linear transformation. The images of the basis vectors for V under φ can be coordinatized relative to S':

$$\varphi(\bar{v}_1) = a_{11}\bar{w}_1 + a_{21}\bar{w}_2 + \dots + a_{m1}\bar{w}_m$$
$$\varphi(\bar{v}_2) = a_{12}\bar{w}_1 + a_{22}\bar{w}_2 + \dots + a_{m2}\bar{w}_m$$
$$\vdots$$
$$\varphi(\bar{v}_n) = a_{1n}\bar{w}_1 + a_{2n}\bar{w}_2 + \dots + a_{mn}\bar{w}_m,$$

so $[\varphi(\bar{v}_1)]_{S'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $[\varphi(\bar{v}_2)]_{S'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, [\varphi(\bar{v}_n)]_{S'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$.

<u>Defn</u>: With notation as above, the <u>matrix of φ relative to bases S, S'</u> is

$$[\varphi]_S^{S'} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[[\varphi(\bar{v}_1)]_{S'}, [\varphi(\bar{v}_2)]_{S'}, \dots, [\varphi(\bar{v}_n)]_{S'} \right]$$

(that is, the jth <u>column</u> of $[\varphi]_S^{S'}$ is the coordinate vector of $\varphi(\bar{v}_j)$ relative to the basis S'.)