The linear transformation $\varphi: V \to W$ can now be described in terms of matrix multiplication (notation as above):

<u>Theorem</u>: If $\varphi: V \to W$ is linear and S, S' is a basis for V, W, respectively, then $$[\varphi]_S^{S'} [\bar{v}]_S = [\varphi(\bar{v})]_{S'}.$$

<u>Proof</u>: If $\bar{v} = \alpha_1 \bar{v}_1 + \cdots + \alpha_n \bar{v}_n$, then by linearity of φ we have $\varphi(\bar{v}) = \alpha_1 \varphi(\bar{v}_1) + \cdots + \alpha_n \varphi(\bar{v}_n)$. Since coordinatization in W is also linear, we have

$$[\varphi(\bar{v})]_{S'} = \alpha_1 [\varphi(\bar{v}_1)]_{S'} + \cdots + \alpha_n [\varphi(\bar{v}_n)]_{S'}$$
$$= \left[[\varphi(\bar{v}_1)]_{S'}, \cdots, [\varphi(\bar{v}_n)]_{S'} \right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = [\varphi(\bar{v})]_S^{S'} [\bar{v}]_S \quad \square$$

<u>Remarks</u>: ① The theorem says that via coordinatization, every linear map between finite dimensional vector spaces can be expressed as matrix multiplication.

② If $V = W$ and $S = S' = B$, $T: V \to V$ a linear operator, we write $[T]_B$ instead of $[T]_S^{S'}$ or $[T]_B^B$, and we call $[T]_B$ the <u>matrix of T relative to B</u>. Then $[T]_B [\bar{v}]_B = [T(\bar{v})]_B$.

<u>EXAMPLES</u>:

① If A is an $n \times n$ matrix and $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is $\varphi = \varphi_A$, then $A = [\varphi]_E$, where E is the standard basis for $\mathbb{R}^n$. (The $j^{\underline{th}}$ column of A is precisely $Ae_j = [\varphi(e_j)]_E$.)

② $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T((x,y,z)) = (x+y, y+z, 3z)$ and $B = \{(1,0,0), (0,1,0), (0,0,1)\}$, the standard basis for $\mathbb{R}^3$. Then

$$\left.\begin{array}{l} T((1,0,0)) = (1,0,0) \\ T((0,1,0)) = (1,1,0) \\ T((0,0,1)) = (0,1,3) \end{array}\right\} \text{so } [T]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{[T]_B} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{[(x,y,z)]_B} = \underbrace{\begin{bmatrix} x+y \\ y+z \\ 3z \end{bmatrix}}_{[T((x,y,z))]_B}$$