(6)

Thearem Let V be a vectorspace over K.
(i) For every $\bar{v} \in V$, $0\bar{v} = \bar{0}$.
(ii) For every $\alpha \in K$, $\alpha\bar{0} = \bar{0}$.

[Note: 0 is the scalar 0 in K; $\bar{0}$ is the zero vector in V.]

Proof [See also Problem 4.2 in text]
(i) Let $\bar{v}$ be a vector in V. Since $0+0=0$ in K, we have
$$0\bar{v} = (0+0)\bar{v} = 0\bar{v} + 0\bar{v} \quad \text{by [M2]}$$
Since $0\bar{v} = \bar{0} + 0\bar{v}$ by [A2], we have by substitution
$$\bar{0} + 0\bar{v} = 0\bar{v} + 0\bar{v}.$$
Now by cancellation, we obtain $\bar{0} = 0\bar{v}$.

(ii) Let $\alpha \in K$ be any scalar. By [A2], $\bar{0} = \bar{0} + \bar{0}$, so
$$\alpha\bar{0} = \alpha(\bar{0}+\bar{0}) = \alpha\bar{0} + \alpha\bar{0} \quad \text{by [M1]}.$$
Again $\bar{0} + \alpha\bar{0} = \alpha\bar{0}$ and we substitute to obtain
$$\bar{0} + \alpha\bar{0} = \alpha\bar{0} + \alpha\bar{0}.$$
Thus $\bar{0} = \alpha\bar{0}$ by cancellation. $\square$

Corollary Let V be a vector space over K.
If $\bar{v}$ is any vector in V, then $-\bar{v} = (-1)\bar{v}$.

Proof: By uniqueness of additive inverses, we need to show that $(-1)\bar{v}$ satisfies the additive inverse axiom [A3]; that is, $(-1)\bar{v} + \bar{v} = \bar{0}$. We have
$$\begin{aligned}
(-1)\bar{v} + \bar{v} &= (-1)\bar{v} + 1\bar{v} && \text{by [M4]} \\
&= (-1+1)\bar{v} && \text{by [M2]} \\
&= 0\bar{v} \\
&= \bar{0} && \text{by the previous thearem.}
\end{aligned}$$
Hence $(-1)\bar{v} + \bar{v} = \bar{0}$ and so $(-1)\bar{v} = -\bar{v}$. $\square$