(EXAMPLES, CONT.)

(3) Let $D: P_3(t) \to P_2(t)$ be the differentiation map, $D(f(t)) = f'(t)$.

Let $S = \{1, t, t^2, t^3\}$, the standard basis for $P_3(t)$, and $S' = \{1+t+t^2, t+t^2, t^2\}$, a basis for $P_2(t)$ [check]. We will find $[D]_S^{S'}$.

First, if $a+bt+ct^2 \in P_2(t)$, then $[a+bt+ct^2]_{S'} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

[End Lec 20] where $x(1+t+t^2) + y(t+t^2) + z(t^2) = a+bt+ct^2$.

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Solving, we obtain $x = a$, $y = b-a$, $z = c-b$ [exercise]. Hence $[a+bt+ct^2]_{S'} = \begin{bmatrix} a \\ b-a \\ c-b \end{bmatrix}$.

Applying D to each of the elements of S, we have

$D(1) = 0$, $D(t) = 1$, $D(t^2) = 2t$, $D(t^3) = 3t^2$

$[D(1)]_{S'} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[D(t)]_{S'} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $[D(t^2)]_{S'} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$, $[D(t^3)] = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$,

hence $[D]_S^{S'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix}$.

Observe that $[\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3]_S = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, hence

$$[D]_S^{S'} [\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3]_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 \\ 2\alpha_2 - \alpha_1 \\ 3\alpha_3 - 2\alpha_2 \end{bmatrix} = [\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2]_{S'} \text{ (check)}$$

$$= [D(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3)]_{S'},$$

as claimed.