Linear Transformations and Matrix Operations

The matrix of a linear transformation is compatible with addition, scalar multiplication, and composition of linear transformations, in the following sense:

Theorem: If F and G are linear transformations from V to W and S, S' is a basis for V, W respectively, then
(i) $[F+G]_S^{S'} = [F]_S^{S'} + [G]_S^{S'}$, and
(ii) $[\alpha F]_S^{S'} = \alpha [F]_S^{S'}$ for $\alpha \in K$.

Theorem: Let $G: V \to W$, $F: W \to U$ be linear transformations, S a basis for V, S' a basis for W, and S'' a basis for U. Then $[F \circ G]_S^{S''} = [F]_{S'}^{S''} \cdot [G]_S^{S'}$.

Thus the matrix of the composite map is the *product* of the matrices for F and G.

The proofs of the theorems are straightforward but tedious. See Problems 6.10, 6.11 for the operator case.

Basis Change and Coordinates

Given two bases S, S' for V, we need to determine how $[\bar{v}]_S, [\bar{v}]_{S'}$ are related for $\bar{v} \in V$ and how $[T]_S, [T]_{S'}$ are related for $T: V \to V$ a linear transformation.

Defn: Let $S = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n\}$ and $S' = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ be bases for V. The matrix

$$P = \left[[\bar{v}_1]_S, [\bar{v}_2]_S, \cdots, [\bar{v}_n]_S \right]$$

is the *change of basis matrix from S to S'*.
The matrix

$$Q = \left[[\bar{u}_1]_{S'}, [\bar{u}_2]_{S'}, \cdots, [\bar{u}_n]_{S'} \right]$$

is the *change of basis matrix from S' to S*.