That is, if

$$\begin{aligned} \bar{v}_1 &= c_{11}\bar{u}_1 + \cdots + c_{n1}\bar{u}_n \\ \bar{v}_2 &= c_{12}\bar{u}_1 + \cdots + c_{n2}\bar{u}_n \\ &\vdots \\ \bar{v}_n &= c_{1n}\bar{u}_1 + \cdots + c_{nn}\bar{u}_n \end{aligned} \quad \text{then } P = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

Observe that P, the change of basis matrix <u>S to S'</u>, is the matrix relative to S of the linear transformation on V sending <u>$\bar{u}_i \in S$ to $\bar{v}_i \in S'$</u>. This is, if $F: V \to V$ is linear with $F(\bar{u}_1) = \bar{v}_1$, $F(\bar{u}_2) = \bar{v}_2, \ldots, F(\bar{u}_n) = \bar{v}_n$, then

$$[F]_S = P.$$

Similarly Q, change of basis matrix <u>S' to S</u> is the matrix relative to S' of the map $V \to V$ sending <u>$\bar{v}_i \in S'$ to $\bar{u}_i \in S$</u>.

<u>Theorem:</u> Let $S = \{\bar{u}_1, \ldots, \bar{u}_n\}$, $S' = \{\bar{v}_1, \ldots, \bar{v}_n\}$,

$P = [[\bar{v}_1]_S, \ldots, [\bar{v}_n]_S]$, $Q = [[\bar{u}_1]_{S'}, \ldots, [\bar{u}_n]_{S'}]$ as above.

For $\bar{w} \in V$:

$$P[\bar{w}]_{S'} = [\bar{w}]_S \text{ and } Q[\bar{w}]_S = [\bar{w}]_{S'}.$$

Moreover, $Q = P^{-1}$.

<u>Proof:</u> See text, Problem 6.22. $\square$

Observe that the theorem holds if $\bar{w}$ is $\bar{u}_i$ or $\bar{v}_i$.

We have $[\bar{v}_i]_{S'} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$ and so $P[\bar{v}_i]_{S'} = i^{\text{th}}$ column of P $= [\bar{v}_i]_S$

Similarly, $[\bar{u}_i]_S = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$ and $Q[\bar{u}_i]_S = i^{\text{th}}$ column of Q $= [\bar{u}_i]_{S'}$

- This observation is a good way to check that you are using the correct basis change matrix.
- Also, the general rule is that the columns of a basis change matrix are the coordinates of the <u>new</u> basis vectors relative to the <u>old</u> basis.