(EXAMPLES, CONT.)

② Let $V = P_1(t)$, $S = \{\bar{u}_1, \bar{u}_2\}$, $\bar{u}_1 = 1+t$, $\bar{u}_2 = 2-t$ and
$S' = \{\bar{v}_1, \bar{v}_2\}$, $\bar{v}_1 = 1+2t$, $\bar{v}_2 = 2+t$.

P = change of basis matrix from S to S'

Verify that $\bar{v}_1 = \frac{5}{3}\bar{u}_1 - \frac{1}{3}\bar{u}_2$ and $\bar{v}_2 = \frac{4}{3}\bar{u}_1 + \frac{1}{3}\bar{u}_2$,

So that $[\bar{v}_1]_S = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$, $[\bar{v}_2]_S = \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}$ and $\boxed{P = \begin{bmatrix} 5/3 & 4/3 \\ -1/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}}$

By the theorem, for $\bar{\omega} \in P_1(t)$, $P[\bar{\omega}]_{S'} = [\bar{\omega}]_S$.

For example, let $\bar{\omega} = 3\bar{v}_1 - 2\bar{v}_2 = 3(1+2t) - 2(2+t) = -1+4t$.

Then
$$P[\bar{\omega}]_{S'} = \frac{1}{3}\begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ -5 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -5/3 \end{bmatrix}.$$

We have $\frac{7}{3}\bar{u}_1 - \frac{5}{3}\bar{u}_2 = \frac{7}{3}(1+t) - \frac{5}{3}(2-t) = -\frac{3}{3} + \frac{12}{3}t = -1+4t = \bar{\omega}$.

Hence $[\bar{\omega}]_S = \begin{bmatrix} 7/3 \\ -5/3 \end{bmatrix} = P[\bar{\omega}]_{S'}$, as expected.

Q = change of basis matrix from S' to S.

To find Q, we can write $\bar{u}_1, \bar{u}_2$ in terms of $\bar{v}_1, \bar{v}_2$, or find $P^{-1} = Q$. Recall that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, so

$$Q = P^{-1} = \frac{1}{\frac{5}{9}+\frac{4}{9}}\begin{bmatrix} 1/3 & -4/3 \\ 1/3 & 5/3 \end{bmatrix} = \boxed{\begin{bmatrix} 1/3 & -4/3 \\ 1/3 & 5/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix} = Q}$$

By the theorem, for $\bar{\omega} \in P_1(t)$, $Q[\bar{\omega}]_S = [\bar{\omega}]_{S'}$

For example, let $\bar{\omega} = 2\bar{u}_1 + 4\bar{u}_2 = 2(1+t) + 4(2-t) = 10-2t$.

Then
$$Q[\bar{\omega}]_S = \frac{1}{3}\begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -14 \\ 22 \end{bmatrix} = \begin{bmatrix} -14/3 \\ 22/3 \end{bmatrix}.$$

We have $-\frac{14}{3}\bar{v}_1 + \frac{22}{3}\bar{v}_2 = -\frac{14}{3}(1+2t) + \frac{22}{3}(2+t) = \frac{30}{3} - \frac{6}{3}t = \bar{\omega}$,

hence
$$[\bar{\omega}]_{S'} = \begin{bmatrix} -14/3 \\ 22/3 \end{bmatrix} = Q[\bar{\omega}]_S, \text{ as expected.} \quad ⌷$$