Basis Change and Linear Operators

Theorem: Let $T: V \to V$ be a linear operator and let $S = \{\bar{u}_1, \cdots, \bar{u}_n\}$, $S' = \{\bar{v}_1, \cdots, \bar{v}_n\}$ be bases for V. If P is the basis change matrix from S to S', then

$$[T]_{S'} = P^{-1}[T]_S P.$$

Proof: For all $\bar{v} \in V$ we have

$$(*) \quad [T]_S [\bar{v}]_S = [T(\bar{v})]_S \text{ and}$$
$$(**) \quad [T]_{S'} [\bar{v}]_{S'} = [T(\bar{v})]_{S'}.$$

Also, for all $\bar{x} \in V$, we have $[\bar{x}]_S = P[\bar{x}]_{S'}$, hence in particular

$$[\bar{v}]_S = P[\bar{v}]_{S'} \text{ and } [T(\bar{v})]_S = P[T(\bar{v})]_{S'}.$$

Hence $(*)$ becomes $[T]_S P[\bar{v}]_{S'} = P[T(\bar{v})]_{S'}$, and so

$$(***) \quad (P^{-1}[T]_S P)[\bar{v}]_{S'} = [T(\bar{v})]_{S'} = [T]_{S'}[\bar{v}]_{S'}$$

for all $\bar{v} \in V$ (by $(**)$).

In particular, this holds for each $\bar{v}_j \in S'$. We have

$[\bar{v}_j]_{S'} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, and if A is any $n \times n$ matrix, $A[\bar{v}_j]_{S'}$ is then the jth column of A. Hence $(***)$ implies that the jth column of $P^{-1}[T]_S P$ is equal to the jth column of $[T]_{S'}$ for each j, and so $P^{-1}[T]_S P = [T]_{S'}$. □

Defn: We say two $n \times n$ matrices A, B are <u>similar</u> if there is an invertible matrix P such that $B = P^{-1}AP$.

The theorem says that matrices of $T: V \to V$ relative to different bases are similar. The next theorem says the converse is true.