<u>Theorem</u>: Two matrices A, B represent the same linear operator relative to (possibly) different bases, if and only if they are similar matrices.

<u>Proof</u>: ($\Rightarrow$) This is the previous theorem.

($\Leftarrow$) Let $B = P^{-1}AP$, P an invertible matrix.
Let $\varphi = \varphi_A : K^n \to K^n$ be the linear map defined by $\varphi(\bar{x}) = A\bar{x}$. Observe that $A = [\varphi]_E$, the matrix of φ relative to the <u>standard</u> basis E for K^n.
Let $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ be the columns of P. Since P is invertible, $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis for K^n. Moreover, P is the basis change matrix from E to S.
Therefore,
$$B = P^{-1}AP = P^{-1}[\varphi]_E P = [\varphi]_S,$$
and so $B = [\varphi]_S$ while $A = [\varphi]_E$. $\square$

Finally, recall that $\det(XY) = \det(X)\cdot\det(Y)$, and $\det(X^{-1}) = 1/\det(X)$. Hence, if $B = P^{-1}AP$, then
$$\det B = \det(P^{-1}AP) = \frac{1}{\det P}(\det A)(\det P) = \det A.$$
That is, similar matrices have the same determinant.

Since any two matrix representation of the same linear operator must be similar, we can define:

<u>Defn</u>: If $T: V \to V$ is a linear operator on a finite dimensional vector space V, the <u>determinant</u> of T is $\det[T]_S$, where S is <u>any</u> basis for V.

[END EXAM II]

End Lec 22