EXAMPLES

(1) Let $A=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}$.

$$\Delta_A(t)=\det(tI-A)=\det\left(\begin{bmatrix}t & 0\\ 0 & t\end{bmatrix}-\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}\right)=\begin{vmatrix}t-1 & 2\\ -3 & t-4\end{vmatrix}$$

$$=(t-1)(t-4)-2(-3)=t^2-5t+4+6$$

$$=t^2-5t+10$$

Observe that $tr(A)=5$ and $\det A=10$.

Also,

$$\Delta_A(A)=A^2-5A+10I=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}-5\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}+10\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=$$

$$=\begin{bmatrix}-5 & -10\\ 15 & 10\end{bmatrix}+\begin{bmatrix}-5 & 10\\ -15 & -20\end{bmatrix}+\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=\begin{bmatrix}-5-5+10 & -10+10\\ 15-15 & 10-20+10\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

These observations are not coincidences and will be explained later.

(2) Let $A=\begin{bmatrix}2 & 0 & 0 & 1\\ 3 & 0 & 1 & 2\\ -1 & 2 & 2 & 1\\ 1 & 0 & 2 & 0\end{bmatrix}$.

(expand along row 1.)

$$\Delta_A(t)=\begin{vmatrix}t-2 & 0 & 0 & -1\\ -3 & t & -1 & -2\\ 1 & -2 & t-2 & -1\\ -1 & 0 & -2 & t\end{vmatrix}=(t-2)\begin{vmatrix}t & -1 & -2\\ -2 & t-2 & -1\\ 0 & -2 & t\end{vmatrix}-(-1)\begin{vmatrix}-3 & t & -1\\ 1 & -2 & t-2\\ -1 & 0 & -2\end{vmatrix}=$$

$$=(t-2)\left[t^2(t-2)-8-2t-2t\right]+\left[-12-t(t-2)+2+2t\right]=$$

$$=(t-2)\left[t^3-2t^2-4t-8\right]+\left[-t^2+4t-10\right]=$$

$$=t^4-2t^3-4t^2-8t-2t^3+4t^2+8t+16-t^2+4t-10$$

$$=\boxed{t^4-4t^3-t^2+4t+6=\Delta_A(t)}$$

Exercise: Verify that $tr(A)=4$, $\det A=6$, $\Delta_A(A)=0$.