Proof of C-H theorem: Let $\Delta_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

Let $B(t)$ be the classical adjoint of $tI - A$, i.e. the transpose of the cofactor matrix [see § 8.9]. Each cofactor of $tI-A$ is a polynomial in t of degree $\leq n-1$. So every entry in $B(t)$ is a polynomial of degree $\leq n-1$.

Let B_k be the $n \times n$ matrix whose (i,j)-entry is the coefficient of t^k in the (i,j)-entry of $B(t)$, so that

$$B(t) = B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_1 t + B_0.$$

By a property of the adjoint (Theorem 8.9), $(tI-A)B(t) = |tI-A| \cdot I$, or

$$(tI - A)(B_{n-1}t^{n-1} + \cdots + B_0) = \Delta_A(t) I = (t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0) I.$$

Multiplying out the left hand side, we obtain

$$\begin{aligned} &B_{n-1}t^n + B_{n-2}t^{n-1} + B_{n-3}t^{n-2} + \cdots + B_1 t^2 + B_0 t \\ &\quad - AB_{n-1}t^{n-1} - AB_{n-2}t^{n-2} - \cdots - AB_2 t^2 - AB_1 t - AB_0 = \\ &\quad = It^n + a_{n-1}I t^{n-1} + \cdots + a_1 I t + a_0 I. \end{aligned}$$

Equating coefficients, we obtain:

$$\begin{array}{rcl} B_{n-1} &=& I \\ B_{n-2} - AB_{n-1} &=& a_{n-1}I \\ B_{n-3} - AB_{n-2} &=& a_{n-2}I \\ &\vdots& \\ B_0 - AB_1 &=& a_1 I \\ -AB_0 &=& a_0 I \end{array} \quad \begin{bmatrix} A^n \\ A^{n-1} \\ A^{n-2} \\ \vdots \\ A^1 \\ A^0 = I \end{bmatrix} \longrightarrow \begin{array}{rcl} \cancel{A^n B_{n-1}} &=& A^n \\ \cancel{A^{n-1}B_{n-2}} - \cancel{A^n B_{n-1}} &=& a_{n-1}A^{n-1} \\ \cancel{A^{n-2}B_{n-3}} - \cancel{A^{n-1}B_{n-2}} &=& a_{n-2}A^{n-2} \\ \vdots && \vdots \\ \cancel{AB_0} - \cancel{A^2 B_1} &=& a_1 A \\ -AB_0 &=& a_0 I. \end{array}$$

Multiply by

Summing the left hand sides of the equations gives 0, and summing the righthand sides gives

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = \Delta_A(A).$$

Hence $\Delta_A(A) = 0$ as claimed. □

End Lec 24 [See Handout for an example illustrating the proof.]