② If λ is <u>not</u> an eigenvalue of T, then we have $\{\bar{v} \in V \mid T(\bar{v}) = \lambda\bar{v}\} = \{\bar{0}\}$, which is also a subspace (but is not an interesting subspace).

<u>Recall</u>: If M is a square matrix, then $\mathcal{N}(M) \neq \{\bar{0}\}$ if and only if $\det M = 0$. (see Theorem 8.5.)

<u>Theorem</u>: Let A be an $n \times n$ matrix over K.
An element λ of K is an eigenvalue of A if and only if λ is a root of the characteristic polynomial $\Delta_A(t)$ of A.
Moreover, if λ is an eigenvalue of A, then $E_\lambda = \mathcal{N}(\lambda I - A)$ and $\dim E_\lambda = \text{nullity}(\lambda I - A)$.

<u>Proof</u>: We have $A\bar{v} = \lambda\bar{v} = (\lambda I)\bar{v}$ if and only if $\lambda I\bar{v} - A\bar{v} = \bar{0}$, That is, if and only if $(\lambda I - A)\bar{v} = \bar{0}$. Hence $A\bar{v} = \lambda\bar{v}$ if and only if $\bar{v}$ is in the null space of $\lambda I - A$.
Now λ is an eigenvalue of A if and only if there is a <u>nonzero</u> vector $\bar{v}$ in $\mathcal{N}(\lambda I - A)$, hence if and only if $\det(\lambda I - A) = \Delta_A(\lambda) = 0$. Thus the eigenvalues of A are precisely the roots of $\Delta_A(t)$.
If λ is an eigenvalue of A, then by the remarks above, $E_\lambda = \{\bar{v} \in V \mid A\bar{v} = \lambda\bar{v}\} = \{\bar{v} \in V \mid (\lambda I - A)\bar{v} = \bar{0}\} = \mathcal{N}(\lambda I - A)$, and so $\dim E_\lambda = \dim \mathcal{N}(\lambda I - A) = \text{nullity}(\lambda I - A)$. $\square$

This translates to linear operators as follows:

<u>Theorem</u> Let $T: V \to V$ be a linear operator.
An element λ of K is an eigenvalue of T if and only if λ is a root of the characteristic polynomial $\Delta_T(t)$ of T.
Moreover, if λ is an eigenvalue of T, then $E_\lambda = \ker(\lambda\mathcal{I} - T)$, where $\mathcal{I}$ is the identity map on V, and $\dim E_\lambda = \text{nullity}(\lambda\mathcal{I} - T)$.