In general, the multiplicity of λ as a root of $\Delta_A(t)$ and $\dim E_\lambda$ are related.

<u>Defn</u>: Let λ be an eigenvalue of the linear operator T. The <u>algebraic multiplicity</u> of λ is the multiplicity of λ as a root of the characteristic polynomial of T. The <u>geometric multiplicity</u> of λ is $\dim E_\lambda$, the dimension of the λ eigenspace.
[Analogous definitions apply to a <u>matrix</u> A.]

<u>Theorem</u>: If $T: V \to V$ is a linear operator with eigenvalue λ, then the geometric multiplicity of λ is less than or equal to the algebraic multiplicity of λ.
That is, if $\Delta_T(t) = (t-\lambda)^\ell f(t)$, $t-\lambda$ not a factor of $f(t)$, then $\dim E_\lambda \leq \ell$.

<u>Proof</u>: Let $r = \dim E_\lambda$, the geometric multiplicity of λ. Let $\{\bar{v}_1, \cdots, \bar{v}_r\}$ be a basis for E_λ, and extend to a basis $S = \{\bar{v}_1, \cdots, \bar{v}_r, \bar{w}_1, \cdots, \bar{w}_m\}$ for V.

Since $T(\bar{v}_j) = \lambda \bar{v}_j$ for all $i = 1, \cdots, r$, $[T(\bar{v}_j)]_S = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ jth position, and so

$$[T]_S = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} \lambda & & O \\ & \lambda & \\ O & & \ddots \\ & & & \lambda \end{matrix} & * \\ \hline O & A \end{array}\right] \begin{matrix} \}\, r \\ \}\, m \end{matrix} \\ \begin{matrix} \underbrace{\qquad\qquad}_{r} & \underbrace{\qquad}_{m} \end{matrix} \end{array}$$

Hence $\Delta_T(t) = (t-\lambda)^r \Delta_A(t)$. Therefore, $(t-\lambda)^r$ is a factor of $\Delta_T(t)$ and so the algebraic multiplicity of λ is <u>at least</u> r. $\square$