Similar Matrices

Recall: Two matrices A, B are similar if $B = P^{-1}AP$ for some invertible matrix P. We showed that similar matrices have the same characteristic polynomial. We therefore have:

Theorem: If A, B are similar matrices, then they have the same eigenvalues.

NOTE: The eigenvectors of A, B are NOT generally the same. They are related as follows:

Theorem: Let A, B be similar matrices with $B = P^{-1}AP$, and let λ be an eigenvalue of A and B.
If $\bar{v}$ is an eigenvector of A belonging to λ, then $P^{-1}\bar{v}$ is an eigenvector of B belonging to λ.

Proof: By definition, $\bar{v} \neq \bar{0}$ and $A\bar{v} = \lambda\bar{v}$.
Since $\bar{v} \neq \bar{0}$ and P^{-1} is nonsingular, we have $P^{-1}\bar{v} \neq \bar{0}$.
Also, $B(P^{-1}\bar{v}) = P^{-1}AP(P^{-1}\bar{v}) = P^{-1}(A\bar{v}) = P^{-1}\lambda\bar{v} = \lambda P^{-1}\bar{v}$.
Hence $P^{-1}\bar{v}$ is an eigenvector of B belonging to λ. $\square$

Remark: Given a matrix A, we would like to find a "nice" matrix B similar to A. This can simplify some matrix calculations.

For example, if $A = P^{-1}BP$, then it is easy to show using induction that $A^n = (P^{-1}BP)^n = P^{-1}B^nP$. [Exercise]
If B is diagonal, for example, then B^n is easy to compute, hence so is $A^n = P^{-1}B^nP$.

$\boxed{\text{End Lec 26}}$