Theorem: An $n \times n$ matrix $A$ is diagonalizable if and only if there is a set of $n$ linearly independent eigenvectors of $A$; that is, if and only if there is a basis $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ for $K^n$ where each $\bar{v}_j$ is an eigenvector of $A$.

In this case, if $P = [\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n]$ is the matrix whose $j^{\underline{th}}$ column is $\bar{v}_j$, then $D = P^{-1}AP$ is diagonal and the $j^{\underline{th}}$ diagonal entry of $D$ is the eigenvalue $\lambda_j$ corresponding to $\bar{v}_j$.

Proof: $\boxed{\Leftarrow}$ Suppose $B = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ is a basis for $V$ such that each $\bar{v}_j$ is an eigenvector of $A$ belonging to $\lambda_j$. Let $\varphi: K^n \to K^n$ be defined by $\varphi(\bar{x}) = A\bar{x}$, so that $A = [\varphi]_E$, for $E$ the standard basis of $K^n$. Let $P = [\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n]$ be the matrix with $\bar{v}_j$ as $j^{\underline{th}}$ column, so that $P$ is the basis change matrix from $E$ to $B$.

We have 
$$D = P^{-1}AP = P^{-1}[\varphi]_E P = [\varphi]_B$$
$$= [[\varphi(v_1)]_B, [\varphi(v_2)]_B, \dots, [\varphi(v_n)]_B]$$

But $\varphi(\bar{v}_j) = \lambda_j \bar{v}_j$, so $[\varphi(\bar{v}_j)]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{\underline{th}}$ position.

Hence $D$ is diagonal with $j^{\underline{th}}$ diagonal entry $\lambda_j$.

$\boxed{\Rightarrow}$ Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ \bigcirc & & \ddots \\ & & & \lambda_n \end{bmatrix}$ be diagonal, and let $\bar{v}_j$ be the $j^{\underline{th}}$ column of $P$. Then $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis for $K^n$ and $P$ is the basis change matrix from $E$ to $B$. Thus if $\varphi = \varphi_A$ as above, then $A = [\varphi]_E$ and $D = P^{-1}AP = [\varphi]_B$. Hence the $j^{\underline{th}}$ column of $D$ is $[\varphi(\bar{v}_j)]_B = [A\bar{v}_j]_B$. But the $j^{\underline{th}}$ column of $D$ is also $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\lambda_j v_j]_B$. Hence $A\bar{v}_j = \lambda_j \bar{v}_j$ for all $j$, and so $B$ is a basis for $K^n$ consisting of eigenvectors. $\square$