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Subspaces (§4.5)

Defn: Let V be a vector space over K. A <u>subspace</u> of V is a subset W of V that is a vector space under the same operations as V.

Theorem (Subspace Criteria) Let V be a vector space over K. A subset W of V is a <u>subspace</u> if and only if

(i) W is non-empty.

(ii) $\bar{w}_1 + \bar{w}_2$ is in W for all $\bar{w}_1, \bar{w}_2 \in W$.
(That is, W is closed under vector addition.)

(iii) $\alpha \bar{w}$ is in W for all $\alpha \in K$, $\bar{w} \in W$.
(That is, W is closed under scalar multiplication)

Proof: $\boxed{\Rightarrow}$ If W is a subspace, then it contains a zero vector by [A2], so (i) holds, and (ii), (iii) follow from [C1] and [C2].

$\boxed{\Leftarrow}$ Assume now that (i), (ii), and (iii) hold. Then [C1] and [C2] hold by (ii), (iii).

Also, [A1], [A4], [M1]–[M4] hold for all vectors <u>in V</u>, hence for all vectors in W.

It remains to show that [A2], [A3] hold.

By (i) there is some vector $\bar{w}$ in W.

By (iii), $0\bar{w} = \bar{0}_V \in W$ and so W has a zero vector and [A2] holds.

Also by (iii), if $\bar{w}$ is any vector in W, then $(-1)\bar{w} = -\bar{w}$ is in W, and since the zero vector of W **is** also $\bar{0}_V$, $-\bar{w}$ is the additive inverse of $\bar{w}$ in W. Hence [A3] holds. $\square$

Note: The empty set $\emptyset$ satisfies both (ii) and (iii), but $\emptyset$ is <u>not</u> a subspace. It is <u>always</u> necessary to show $W \neq \emptyset$.