We have shown that a matrix A over K, or linear operator $T: V \to V$, is diagonalizable if and only if K^n, respectively V, has a basis consisting of eigenvectors. The following result allows us to check one eigenvalue at a time.

Theorem: Let $T: V \to V$ be a linear operator. If $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues of T and $\bar{u}_i$ is an eigenvector belonging to λ_i, then $\{u_1, u_2, \dots, u_r\}$ is linearly independent.

Proof: Proceed by induction on r.
Since $\bar{u}_1 \neq \bar{0}$, $\{\bar{u}_1\}$ is linearly independent, so the statement is true when $r=1$.
Assume now that a set of $r-1$ eigenvectors belonging to distinct eigenvalues is linearly independent.
Suppose $(*)$ $\alpha_1 \bar{u}_1 + \cdots + \alpha_{r-1} \bar{u}_{r-1} + \alpha_r \bar{u}_r = \bar{0}$.
Applying T to both sides and using linearity of T, we have

$$\alpha_1 T(\bar{u}_1) + \cdots + \alpha_{r-1} T(\bar{u}_{r-1}) + \alpha_r T(\bar{u}_r) = T(\bar{0}) = \bar{0}.$$

Since $\bar{u}_i$ is an eigenvector belonging to λ_i for each i, we get

$$(**) \quad \alpha_1 \lambda_1 \bar{u}_1 + \cdots + \alpha_{r-1} \lambda_{r-1} u_{r-1} + \alpha_r \lambda_r u_r = \bar{0}.$$

Multiply $(*)$ by λ_r to obtain

$$\alpha_1 \lambda_r \bar{u}_1 + \cdots + \alpha_{r-1} \lambda_r \bar{u}_{r-1} + \alpha_r \lambda_r \bar{u}_r = \bar{0},$$

and subtract this from $(**)$ to obtain

$$\alpha_1 (\lambda_1 - \lambda_r) u_1 + \cdots + \alpha_{r-1} (\lambda_{r-1} - \lambda_r) u_{r-1} = \bar{0}$$

By the inductive hypothesis, $\{\bar{u}_1, \dots, \bar{u}_{r-1}\}$ is linearly independent, hence $\alpha_i(\lambda_i - \lambda_r) = 0$ for all $i = 1, \dots, r-1$.
Since the eigenvalues are all distinct, $\lambda_i - \lambda_r \neq 0$, hence $\underline{\alpha_i = 0 \text{ for } i = 1, \dots, r-1}$. Then $(*)$ becomes $\alpha_r \bar{u}_r = \bar{0}$ and since $\bar{u}_r \neq \bar{0}$, this implies $\underline{\alpha_r = \bar{0}}$ as well. Therefore $\{\bar{u}_1, \dots, \bar{u}_{r-1}, \bar{u}_r\}$ is linearly independent.

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The theorem follows by induction. □