EXAMPLES:

① Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$, so that $tI - A = \begin{bmatrix} t-1 & -1 & -2 \\ 0 & t-2 & -3 \\ 0 & 0 & t-3 \end{bmatrix}$,

and $\Delta_A(t) = (t-1)(t-2)(t-3)$. There are 3 distinct eigenvalues, so A is diagonalizable.

Find bases for the eigenspaces for $\lambda=1$, $\lambda=2$ and $\lambda=3$:

E_1: $1I - A = \begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -3 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ x free, $y + 2z = 0$, $z = 0$, so $y = 0$

Therefore, a basis for E_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

E_2: $2I - A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ y free, $z = 0$, $x - y - 2z = 0$, so $x = y + 2z = y$.

Therefore, a basis for E_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

E_3: $3I - A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ z free, $y - 3z = 0$, $y = 3z$, $2x - y - 2z = 0$, $x = \frac{1}{2}y + z = \frac{3}{2}z + z = \frac{5}{2}z$.

Letting $z = 2$, a basis for E_3 is $\left\{ \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix} \right\}$.

Hence $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$ of eigenvectors, belonging to $\lambda = 1$, $\lambda = 2$, $\lambda = 3$, respectively.

Therefore, if $P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ then $P A P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (check!)

$\left(P^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix} \right)$

End Lec 28