<u>Theorem:</u> Let A be a square matrix. If $f(t)$ is any polynomial such that $f(A)=0$, then $m(t)$ is a factor of $f(t)$.

<u>Proof:</u> Let $f(A)=0$. By the Division Algorithm,
$f(t)=Q(t)\,m(t)+R(t)$ with $R(t)=0$ or $\deg R(t)<\deg m(t)$.
We have (by theorem 9.1):

$$0=f(A)=Q(A)\,m(A)+R(A)=Q(A)\cdot 0+R(A)=R(A).$$

Hence $R(A)=0$. By the minimality of $\deg m(t)$, we cannot have $R(t)\neq 0$ and $\deg R(t)<\deg m(t)$, hence $R(t)=0$, and so $f(t)=Q(t)\,m(t)$. □

This theorem has several important consequences:

<u>Corollary:</u> The minimal polynomial of a matrix A is unique.

<u>Proof:</u> If there are two <u>monic</u> polynomials $m_1(t)$, $m_2(t)$ of (the same) smallest degree with $m_1(A)=m_2(A)=0$, then $m_2(t)=Q(t)\,m_1(t)$ for some $Q(t)$. But since $\deg m_1(t)=\deg m_2(t)$, we have that $Q(t)=C$ is a constant. Hence $m_2(t)=C\,m_1(t)$, and since both are monic, we have $C=1$. Therefore $m_1(t)=m_2(t)$. □

<u>Corollary:</u> If A is a square matrix, then $m_A(t)$ is a factor of $\Delta_A(t)$.

More precisely (see Problems 9.34, 9.35):

<u>Theorem:</u> If A is a square matrix, then the <u>irreducible factors</u> of $m_A(t)$ and $\Delta_A(t)$ are the same.

<u>Corollary:</u> If A is a square matrix, then the eigenvalues of A are precisely the roots of $m_A(t)$.