Hence, if $\Delta_A(t)$ factors as
$$\Delta_A(t) = (t-\lambda_1)^{k_1}(t-\lambda_2)^{k_2}\cdots(t-\lambda_\ell)^{k_\ell},$$
where the λ_i are distinct and $k_i \geq 1$ for all i, then
$$m_A(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_\ell)^{m_\ell},$$
where $1 \leq m_i \leq k_i$ for all i.

EXAMPLES:

① If A is an $n\times n$ matrix and $A^k = 0$ for some positive integer k, then $\Delta_A(t) = t^n$. [we say A is nilpotent in this case]

Proof: Since $A^k = 0$, A is a zero of $f(t) = t^k$. Hence $m(t)$ is a factor of t^k, so $m(t) = t^m$ for some $m \geq 1$.
Now the only irreducible factor of $m(t)$, hence of $\Delta_A(t)$ is t, and since $\deg \Delta_A(t) = n$, we have $\Delta_A(t) = t^n$. □

Note: $A^k = 0$ does not imply that $A = 0$.

② If $A^2 = -I$, then A has no real number eigenvalues.

Proof: Since $A^2 = -I$, $A^2 + I = 0$, and so A is a zero of $f(t) = t^2+1$. Hence $m(t)$ is a factor of t^2+1 and all roots of $m(t)$, thus eigenvalues of A, are roots of t^2+1.
End Lec 29 But t^2+1 has no real roots, so A has no real eigenvalues. □

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③ What are the possible minimal polynomials for a matrix A with characteristic polynomial $\Delta_A(t) = (t-4)^3(t-5)^2$?

We know A is 5×5 and $m_A(t) = (t-4)^\ell (t-5)^m$, with $1 \leq \ell \leq 3$, $1 \leq m \leq 2$. There are six possibilities (and all can occur):

$(t-4)(t-5)$

$(t-4)^2(t-5)$, $(t-4)(t-5)^2$

$(t-4)^3(t-5)$ $(t-4)^2(t-5)^2$

$(t-4)^3(t-5)^2$.