Further Results on Minimal Polynomials

Theorem: If A, B are similar matrices, then $\Delta_A(t) = \Delta_B(t)$ and $m_A(t) = m_B(t)$.

We need the following exercise for the proof:

Exercise: If $B = P^{-1}AP$ then $B^k = P^{-1}A^kP$ for all positive integers k. Moreover, if $f(t)$ is a polynomial, then $f(B) = P^{-1}f(A)P$; that is $f(P^{-1}AP) = P^{-1}f(A)P$.

Proof of theorem: We have already shown that $\Delta_A(t) = \Delta_B(t)$. For the minimal polynomials, note that by the Exercise above, $f(A) = 0$ if and only if $f(B) = 0$. Hence A, B are zeros of precisely the same polynomials, and so by definition of minimal polynomial, we have $m_A(t) = m_B(t)$. $\square$

Remark: The converse of theorem is false. Two matrices can have the same minimal and characteristic polynomials without being similar. For example,

$$A = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|c|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

both have $\Delta(t) = (t-2)^4$ and $m(t) = (t-2)^2$ [check!], but A, B are not similar. [This is not easy to prove with our results so far.]

Theorem: A matrix A is diagonalizable over K if and only if $m_A(t)$ factors over K as a product of distinct linear factors.

Proof: $\Rightarrow$ Exercise.

[End Lec 30] $\Leftarrow$ Proof will be given later. $\square$