Invariant Subspaces (§10.3)

We continue toward our goal of constructing a basis B for V so that for a given linear operator $T: V \to V$, the matrix $[T]_B$ is as "nice" as possible. [A first approximation, beyond diagonal, is triangular form. READ §10.2.]

The next step is to consider certain "invariant" subspaces.

Defn: Let $T: V \to V$ be a linear operator. A subspace W of V is T-invariant (or invariant under T) if $T(\bar{w}) \in W$ for all $\bar{w} \in W$. (That is, $T(W) \subseteq W$.)

Note: This does NOT say that each vector in W is fixed by T.

Defn: Let $T: V \to V$ be a linear operator, W a T-invariant subspace. The restriction of T to W is the map $\hat{T}: W \to W$ defined by $\hat{T}(\bar{w}) = T(\bar{w})$ for all $\bar{w} \in W$. We sometimes denote $\hat{T}$ by T_W.

Note: Since T is invariant, $\hat{T}$ is a map from W to W. It is an easy exercise to show that $\hat{T}$ is linear.

Theorem: Let $T: V \to V$ be a linear operator and W a T-invariant subspace. There is a basis S for V such that $[T]_S = \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]$, where A is a matrix representation of $\hat{T} = T_W$

Proof: Let $\{\bar{w}_1, \dots, \bar{w}_m\} = S_0$ be a basis for W, and extend to a basis $\{\bar{w}_1, \dots, \bar{w}_m, \bar{v}_1, \dots \bar{v}_r\} = S$ for V. Observe that since each w_j is in W, we have $T(\bar{w}_j) \in W$, so is a linear combination of S_0.

For $j = 1, \dots, m$, let $T(\bar{w}_j) = a_{1j}w_1 + \dots + a_{mj}w_m = a_{1j}\bar{w}_1 + \dots + a_{mj}\bar{w}_m + 0\bar{v}_1 + \dots + 0\bar{v}_r$.

Hence $[\hat{T}(\bar{w}_j)]_{S_0} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$ and $[T(\bar{w}_j)]_S = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and if $A = (a_{ij})$, then $[\hat{T}]_{S_0} = A$,

and $[T]_S = \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]$, where O is the $r \times r$ zero matrix. $\square$