EXAMPLES:

In all examples, $T: V \to V$ is a linear operator.

① $\{\bar{0}_V\}$ and V are clearly T-invariant.

② $\operatorname{Im} T$ is T-invariant.

Proof: If $\bar{v} \in \operatorname{Im} T$, then $T(\bar{v}) \in \operatorname{Im} T$ as well. (Of course, if $\bar{v}$ is any vector in V, then $T(\bar{v}) \in \operatorname{Im} T$.) □

③ $\ker T$ is T-invariant.

Proof: If $\bar{v} \in \ker T$ then $T(\bar{v}) = \bar{0}_V$, and $\bar{0}_V \in \ker T$. □

④ If λ is an eigenvalue of T, then E_λ is T-invariant.

Proof: We need to show that if $\bar{\omega} \in E_\lambda$, then $T(\bar{\omega}) \in E_\lambda$. By definition of E_λ, if $\omega \in E_\lambda$ then $T(\bar{\omega}) = \lambda\bar{\omega}$. Since E_λ is a subspace and $\lambda \in K$, $\bar{\omega} \in E_\lambda$, we know $\lambda\bar{\omega} \in E_\lambda$. Hence $T(\bar{\omega}) \in E_\lambda$ as claimed. □

⑤ If $f(t)$ is any polynomial over K, then $\ker f(T)$ is T-invariant.

Proof: Recall (§9.2) that polynomials in the same linear operator commute. In particular, $T \circ f(T) = f(T) \circ T$.

We need to show that if $\bar{\omega} \in \ker f(T)$, then $T(\bar{\omega}) \in \ker f(T)$. We have

$$f(T)(T(\bar{\omega})) = (f(T) \circ T)(\bar{\omega}) = (T \circ f(T))(\bar{\omega}) = T(f(T)(\bar{\omega})) = T(\bar{0}_V) = \bar{0}_V,$$

hence $T(\bar{\omega}) \in \ker f(T)$ as claimed. □

⑥ A one dimensional subspace $W = \operatorname{sp}\{\bar{\omega}\}$ is T-invariant if and only if $\bar{\omega}$ is an eigenvector for some eigenvalue λ of T.

Proof: Exercise. □

Defn: Let $T: V \to V$ be a linear operator and let λ be an eigenvalue of T with algebraic multiplicity m. The generalized eigenspace of T belonging to λ is $K_\lambda = \ker((T - \lambda I)^m)$.

[For an $n \times n$ matrix A, $K_\lambda = \mathcal{N}((A - \lambda I)^m)$.]