Remarks:

① Since $K_\lambda = \ker f(T)$, where $f(t) = (t-\lambda)^m$, K_λ is a T-invariant subspace of V by example ⑤.

② $E_\lambda = \ker(T-\lambda I) \subseteq \ker(T-\lambda I)^2 \subseteq \ker(T-\lambda I)^3 \subseteq \cdots \subseteq \ker(T-\lambda I)^m = K_\lambda$ (HW problem).
Moreover, $\ker(T-\lambda I)^j = \ker(T-\lambda I)^m$ for all $j \geq m$ (and, in fact, for all $j \geq l$, where l is the multiplicity of λ as a root of $m_T(t)$).

③ We will show later that $\dim K_\lambda$ is equal to the algebraic multiplicity of λ and that a union of the bases for all generalized eigenspaces is a basis B for V. With a proper choice of these bases, $[T]_B$ will be as close to diagonal as possible. (Note $[T]_B$ will be diagonal if and only if $K_\lambda = E_\lambda$ for every eigenvalue λ of T.)

End Lec 31

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④ We can first choose a basis for $E_\lambda = \ker(T-\lambda I)$, extend to a basis for $\ker(T-\lambda I)^2$, etc., until we obtain a basis B_λ for K_λ.

Observe that if $\bar{v} \in \ker(T-\lambda I)^k$, then $(T-\lambda I)(\bar{v})$ is in $\ker(T-\lambda I)^{k-1}$. That is, $T(\bar{v}) - \lambda\bar{v} = \bar{w} \in \ker(T-\lambda I)^{k-1}$. Hence $T(\bar{v}) = \lambda\bar{v} + \bar{w}$, where $w \in \ker(T-\lambda I)^{k-1}$. It follows that if $\bar{v}$ is one of the basis vectors in B_λ, then $T(\bar{v}) = \lambda\bar{v} + \bar{w}$, where $\bar{w}$ is a sum of previous vectors in B_λ. Hence $[T_{K_\lambda}]_{B_\lambda}$ is upper triangular, with λ on the diagonal.

[Choosing such a basis for each eigenvalue and taking B to be the union of these bases, we have that B is a basis for V and $[T]_B$ is block diagonal with one such block for each λ.]