## EXAMPLE:

Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & -2 & 0 & 0 & -6 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ -1 & 2 & 0 & 0 & 5 \end{bmatrix}$, $T = \varphi_A : \mathbb{R}^5 \to \mathbb{R}^5$, so that $\Delta_A(t) = \Delta_T(t) = (t-1)^3(t-2)^2$.

Find a basis for $K_1$ as described above:

$E_1 = \mathcal{N}(A - 1I)$.

$$A - 1I = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 2 & -3 & 0 & 0 & -6 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ -1 & 2 & 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $E_1$ is the set of $\bar{v} = \begin{bmatrix} x \\ y \\ z \\ r \\ s \end{bmatrix}$ with $(A - 1I)\bar{v} = \bar{0}$,

So $\underline{s \text{ is free}}$, $\underline{z = r = 0}$, $\underline{y = -2s}$, $x = 2y + 4s = -4s + 4s = \underline{0 = x}$.

Hence $E_1 = \left\{ \begin{bmatrix} 0 \\ -2s \\ 0 \\ 0 \\ s \end{bmatrix} \middle| \; s \in \mathbb{R} \right\}$, and $\boxed{\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } E_1.}$

Extend to a basis for $\ker(T - 1I)^2 = \mathcal{N}(A - 1I)^2$:

$$(A - 1I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 4 & 2 & -6 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & -2 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 1/2 & 2 \\ 0 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\underline{x, s \text{ are free}}$, $\underline{r = 0}$, so $\underline{z = 0}$, and $\underline{y = -2s}$, and so

$\ker(T - 1I)^2 = \left\{ \begin{bmatrix} x \\ -2s \\ 0 \\ 0 \\ s \end{bmatrix} \middle| \; x, s \in \mathbb{R} \right\}$, and $\boxed{\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}}$ is a basis.