Extend to a basis for $K_1 = \ker(T-1I)^3 = \mathcal{N}(A-1I)^3$.

$$(A-1I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & -6 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x, r, s$ are free, $z = -r$, $y = -2s$, and so

$$K_1 = \left\{ \begin{bmatrix} x \\ -2s \\ -r \\ r \\ s \end{bmatrix} \middle| \; x, r, s \in \mathbb{R} \right\} \text{ and } \boxed{\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}} \text{ is a basis.}$$

In $\ker(T-1I)$

In $\ker(T-1I)^2$

All are in $\ker(T-1I)^3 = K_1$.

## Basis for $K_2$

$E_2 = \ker(T-2I) = \mathcal{N}(A-2I)$.

$$A-2I = \begin{bmatrix} -1 & 0 & 2 & 1 & 0 \\ 2 & -4 & 0 & 0 & -6 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ -1 & 2 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $r, s$ are free, $z = -\frac{1}{2}r$, $y = z + \frac{1}{2}r - \frac{3}{2}s = -\frac{3}{2}s = y$,
$x = 2z + r = -r + r = 0 = x$.

Therefore,

$$E_2 = \left\{ \begin{bmatrix} 0 \\ -\frac{3}{2}s \\ -\frac{1}{2}r \\ r \\ s \end{bmatrix} \middle| \; r, s \in \mathbb{R} \right\}, \text{ and a basis is } \boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}}$$

Since $\dim E_2 =$ algebraic multiplicity of 2, we have $E_2 = K_2$ (verify).

$$\beta = \left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^5.$$