Direct Sum Decompositions (§10.4)

Recall: A vector space V is a direct sum of subspaces $W_1, W_2, \dots, W_m$, written $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$, if
(i) $V = W_1 + W_2 + \cdots + W_m$, and
(ii) $W_i \cap (W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_m) = \{0_V\}$ for all i.

Equivalently, $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ if and only if every vector $\bar{v}$ in V can be written in exactly one way in the form $\bar{v} = \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_m$, with $\bar{w}_i \in W_i$ for each i.

Lemma: Let V be a vector space with subspaces $W_1, \dots, W_m$, and let B_i be a basis for W_i for $i = 1, \dots, m$. Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ if and only if $B = B_1 \cup B_2 \cup \cdots \cup B_m$ is a basis for V.

Proof: See text, Problem 10.7. □
[Essentially (i) holds if and only if B spans V and (ii) holds if and only if B is linearly independent.]

Recall that if $T: V \to V$ is a linear operator and W is a T-invariant subspace, we can find a basis $S_0 = \{\bar{w}_1, \dots, \bar{w}_m\}$ for W and extend to a basis $S = \{\bar{w}_1, \dots, \bar{w}_m, \bar{v}_1, \dots, \bar{v}_r\}$ for V.
If $W' = sp\{\bar{v}_1, \dots, \bar{v}_r\}$, then by the Lemma, $V = W \oplus W'$. However, W' need not be T-invariant.
We showed $[T]_S = \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]$, where $A = [\hat{T}]_{S_0}$, ($\hat{T}$ the restriction of T to W).
If W' is T-invariant, we can say more:

Theorem: If T is a linear operator on V and $V = W \oplus W'$, where W, W' are both T-invariant, with bases S, S', respectively, then $B = S \cup S'$ is a basis for V and $[T]_B = \left[\begin{array}{c|c} A & O \\ \hline O & C \end{array}\right]$, where $A = [T_W]_S$, $C = [T_{W'}]_{S'}$.