Proof: Let $S = \{\bar{w}_1, \ldots, \bar{w}_m\}$, $S' = \{\bar{v}_1, \ldots, \bar{v}_r\}$ be bases for W, W'. Since $V = W \oplus W'$, the Lemma says $B = S \cup S'$ is a basis for V.

Both W and W' are T-invariant, so for each i we have $T(\bar{w}_i) \in W$ and for each j we have $T(\bar{v}_j) \in W'$. Hence

$$T(\bar{w}_i) = a_{1i}\bar{w}_1 + \cdots + a_{mi}\bar{w}_m$$
$$= a_{1i}\bar{w}_1 + \cdots + a_{mi}w_m + 0\bar{v}_1 + \cdots + 0\bar{v}_r, \text{ and}$$
$$T(v_j) = c_{1j}\bar{v}_1 + \cdots + c_{rj}\bar{v}_r$$
$$= 0\bar{w}_1 + \cdots + 0\bar{w}_m + c_{1j}\bar{v}_1 + \cdots + c_{rj}\bar{v}_r.$$

Hence, setting $A = (a_{ki})$ and $C = (c_{kj})$, we have $[T_W]_S = A$, $[T_{W'}]_{S'} = C$ and $[T]_B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right]$. □

Theorem: Let T be a linear operator on V. If $V = W_1 \oplus W_2$ where W_1, W_2 are T-invariant, and $T_1 = T_{W_1}$, $T_2 = T_{W_2}$, then

(i) $\Delta_T(t) = \Delta_{T_1}(t)\Delta_{T_2}(t)$, and

(ii) $m_T(t) = \operatorname{lcm}\{m_{T_1}(t), m_{T_2}(t)\}$.

Proof: By the previous theorem, we can find a basis B such that $[T]_B = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$, where A_i is the matrix for T_i relative to some basis for W_i.

By Theorem 9.18, we have

$$\Delta_T(t) = \Delta_{[T]_B}(t) = \Delta_{A_1}(t)\Delta_{A_2}(t) = \Delta_{T_1}(t)\Delta_{T_2}(t).$$

By Theorem 9.19, we have

$$m_T(t) = m_{[T]_B}(t) = \operatorname{lcm}\{m_{A_1}(t), m_{A_2}(t)\}$$
$$= \operatorname{lcm}\{m_{T_1}(t), m_{T_2}(t)\}. \quad □$$