Degree Graphs of Simple Orthogonal and Symplectic Groups

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ABSTRACT

Let G be a finite group and let $\mathrm{cd}(G)$ be the set of irreducible ordinary character degrees of G. The degree graph of G is the graph $\Delta(G)$ whose set of vertices is the set of primes dividing degrees in $\mathrm{cd}(G)$, with an edge between primes p and q if pq divides some degree in $\mathrm{cd}(G)$. We determine the graph $\Delta(G)$ for the finite simple groups of types B_ℓ, C_ℓ, D_ℓ and ${}^2D_\ell$; that is, for the simple orthogonal and symplectic groups.

1. INTRODUCTION

A problem of interest in the character theory of finite groups is to determine information that can be deduced about the structure of a finite group G from its set of irreducible ordinary character degrees. One tool that has been used to study the relationship between the structure of G and its set of character degrees is the character degree graph $\Delta(G)$.

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G. Denote the set of irreducible character degrees of G by $\mathrm{cd}(G) = \{\chi(1) | \chi \in \mathrm{Irr}(G)\}$ and let $\rho(G)$ denote the set of primes dividing degrees in $\mathrm{cd}(G)$. The character degree graph of G is the graph $\Delta(G)$ whose set of vertices is $\rho(G)$, with primes p, q in $\rho(G)$ joined by an edge if pq divides a for some character degree $a \in \mathrm{cd}(G)$.

The structure of this graph was first studied in the case where G is a solvable group (see [15], [16], and [10], for example). Recently, Lewis and the author have obtained some results on the structure of $\Delta(G)$ for arbitrary finite groups, essentially by using known results for solvable groups to reduce the problem to the structure of $\Delta(G)$ for a finite simple group G

(see [11] and [12]). It is therefore very useful to have as much information about the graphs for the finite simple groups as possible. For example, we obtained a bound on the diameter of $\Delta(G)$ in [12] using the fact proved there that the only simple group whose graph is of diameter three is the sporadic simple Janko group J_1. We hope to improve on that bound, and for this it will be necessary to know precisely which simple groups have graphs of diameter two.

The character tables of the sporadic simple groups are known (see the Atlas [4]) and the graphs for these groups are easily determined. These graphs are described briefly in [12]. The graphs for the alternating groups can be found using the Atlas character tables and the results of [1]. The graphs for the simple groups of exceptional Lie type and for the simple linear and unitary groups were determined by the author in [20] and [21], respectively.

By the Classification of Finite Simple Groups, this leaves the graphs for the simple groups of Lie type of the types B_ℓ, C_ℓ, D_ℓ and ${}^2D_\ell$, that is, the simple orthogonal and symplectic groups, to be determined. The purpose of this paper is to complete the description of the degree graphs of the finite simple groups by proving the following theorem.

THEOREM 1.1. *If G is a finite simple group of Lie type of one of the types B_ℓ with $\ell \geqslant 2$, C_ℓ with $\ell \geqslant 3$, D_ℓ with $\ell \geqslant 4$, or ${}^2D_\ell$ with $\ell \geqslant 4$, then $\Delta(G)$ is a complete graph.*

The restrictions on ℓ are due to isomorphisms of these groups with other groups of Lie type already considered when ℓ is smaller. The groups considered in the theorem are the orthogonal group $\Omega_{2\ell+1}(q)$, the symplectic group $\mathrm{PSp}_{2\ell}(q)$, and the orthogonal groups $\mathrm{P}\Omega^+_{2\ell}(q)$, $\mathrm{P}\Omega^-_{2\ell}(q)$, respectively. (In Atlas [4] notation, these are the groups $\mathrm{O}_{2\ell+1}(q)$, $\mathrm{S}_{2\ell}(q)$, $\mathrm{O}^+_{2\ell}(q)$, and $\mathrm{O}^-_{2\ell}(q)$, respectively.)

Theorem 1.1 and the results of [11], [12], [20], and [21] imply the following corollary.

COROLLARY 1.2. *Let G be a finite simple group. The graph $\Delta(G)$ is disconnected if and only if $G \cong \mathrm{PSL}_2(q)$ for some prime power q. If $\Delta(G)$ is connected, then the diameter of $\Delta(G)$ is at most 3, and we have the following.*

1. *The diameter of $\Delta(G)$ is 3 if and only if $G \cong J_1$.*
2. *The diameter of $\Delta(G)$ is 2 if and only if G is isomorphic to one of*
   (a) *the sporadic Mathieu group M_{11} or M_{23},*
   (b) *the alternating group A_8,*
   (c) *the Suzuki group ${}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geqslant 1$,*
   (d) *the linear group $\mathrm{PSL}_3(q)$, where $q > 2$ is even or q is odd and $q-1$ is divisible by a prime other than 2 or 3, or*

(e) the unitary group $\mathrm{PSU}_3(q)$*, where* $q > 2$ *and* $q+1$ *is divisible by a prime other than* 2 *or* 3.

Otherwise, $\Delta(G)$ *is a complete graph.*

2. CHARACTER DEGREES

In this section, we describe the character degrees used in the proof of Theorem 1.1. We denote by q a power of a prime p, $\mathbb{F}_{q^a}$ is the field of q^a elements, and $\mathbb{F}_{q^a}^*$ its multiplicative group. An algebraic closure of the field $\mathbb{F}_p$ of p elements will be denoted by $\overline{\mathbb{F}}_p$. We will denote by Φ_j the value of the jth cyclotomic polynomial evaluated at q.

For notation, definitions, and basic properties of groups of Lie type, we refer to [2] or [5]. We will denote by $\boldsymbol{G}$ a simple linear algebraic group of adjoint type defined over $\overline{\mathbb{F}}_p$, and by F a Frobenius endomorphism of $\boldsymbol{G}$ so that the set $G = \boldsymbol{G}^F$ of fixed points is finite and the derived group L of $\boldsymbol{G}^F$ is a finite simple group. Let $(\boldsymbol{G}^*, F^*)$ denote the dual of $(\boldsymbol{G}, F)$, and let $G^* = \boldsymbol{G}^{*F^*}$.

Many of the character degrees will be determined using the following lemma, which is a direct result of [5, Theorem 13.23, Remark 13.24] or [2, §12.9] and is also stated in [21, Lemma 2.1].

Lemma 2.1. *There is a bijection between the set of conjugacy classes* (s) *of semisimple elements* s *of* $\boldsymbol{G}^{*F^*}$ *and the set of geometric conjugacy classes* $\mathcal{E}(\boldsymbol{G}^F, (s))$ *of irreducible characters of* $\boldsymbol{G}^F$*. For a semisimple element* s *of* $\boldsymbol{G}^{*F^*}$*, there is a bijection* ψ_s *between the set of irreducible characters in* $\mathcal{E}(\boldsymbol{G}^F, (s))$ *and the set of unipotent characters of* $C_{\boldsymbol{G}^*}(s)^{F^*}$*. Moreover, for* $\chi \in \mathcal{E}(\boldsymbol{G}^F, (s))$*, the degree of* χ *is*

$$\chi(1) = \frac{|\boldsymbol{G}^F|_{p'}}{|C_{\boldsymbol{G}^*}(s)^{F^*}|_{p'}} \psi_s(\chi)(1).$$

The *semisimple character* corresponding to the conjugacy class of s is the character χ_s such that $\psi_s(\chi_s)$ is the principal character of $C_{\boldsymbol{G}^*}(s)^{F^*}$. The lemma says that the irreducible characters of $G = \boldsymbol{G}^F$ are in bijection with the set of pairs (χ_s, μ_s), where χ_s is the semisimple character corresponding to (s) and μ_s is a unipotent character of $C_{\boldsymbol{G}^*}(s)^{F^*}$. The degree of the character corresponding to (χ_s, μ_s) is $\chi_s(1)\mu_s(1)$.

The semisimple elements of $\boldsymbol{G}^{*F^*}$ and their centralizers are found using the results of [3]. The corresponding semisimple character degrees are then computed using the results in [3, §8] or Lemma 2.1. The degrees of the unipotent characters of the classical groups are computed using formulas found in [2, §13.8]. The degree of a general irreducible character corresponding to the pair (χ_s, μ_s) is then found using Lemma 2.1.

LEMMA 2.2. *If $G \cong B_\ell(q)$ is of adjoint type with $\ell \geqslant 3$, then G has irreducible characters χ^α, χ_c, and χ_1 with degrees*

$$\begin{aligned}
\chi^\alpha(1) &= \frac{1}{2}q^4\frac{(q^{\ell-2}-1)(q^{\ell-1}-1)(q^{\ell-1}+1)(q^\ell+1)}{(q^2-1)^2} \\
\chi_c(1) &= \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^\ell+1} \\
&= (q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1) \\
\chi_1(1) &= q\frac{(q^4-1)(q^6-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell-1}+1}
\end{aligned}$$

Proof. In this case, $G \cong \mathrm{SO}_{2\ell+1}(q)$ and the dual group is $G^* \cong \mathrm{Sp}_{2\ell}(q)$. The character χ^α is the unipotent character of G corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & & 2 & & \ell \\ & 0 & & 1 & \end{pmatrix}.$$

The degree $\chi^\alpha(1)$ can be computed using the formula in [2, §13.8].

The character χ_c of G is the semisimple character corresponding to the conjugacy class of a regular element s in the Coxeter torus of G^*, as in [11]. Such an element exists by [19, Theorem 1]. Explicitly, let η be a generator of $\mathbb{F}^*_{q^{2\ell}}$ and let $\tau = \eta^{q^\ell-1}$ be of order $q^\ell+1$. The semisimple element s is conjugate in $\boldsymbol{G}^*$ to the matrix

$$\boldsymbol{s} = \mathrm{diag}[\tau, \tau^{-1}, \tau^q, \tau^{-q}, \tau^{q^2}, \tau^{-q^2}, \ldots, \tau^{q^{\ell-1}}, \tau^{-q^{\ell-1}}].$$

In the notation of [3], this conjugacy class corresponds to the pair of partitions $\lambda = (1^\ell)$, $\mu = \varnothing$ with $\eta^{(1)} = (\ell)$ and $\xi^{(1)} = \varnothing$. By [3, §8], the centralizer in G^* is of order $q^\ell+1$ and the degree of χ_c is as claimed.

The character χ_1 is obtained as follows. Let θ be a generator of $\mathbb{F}^*_{q^{2(\ell-1)}}$ and let $\gamma = \theta^{q^{\ell-1}-1}$ be of order $q^{\ell-1}+1$. Let $s_1 \in G^*$ be conjugate in $\boldsymbol{G}^*$ to

$$\boldsymbol{s_1} = \mathrm{diag}[1, 1, \gamma, \gamma^{-1}, \gamma^q, \gamma^{-q}, \gamma^{q^2}, \gamma^{-q^2}, \ldots, \gamma^{q^{\ell-2}}, \gamma^{-q^{\ell-2}}].$$

Thus s_1 corresponds to a regular element in the Coxeter torus of $\mathrm{Sp}_{2(\ell-1)}(q)$. Such a regular element exists for all q by [19], [7], or direct computation since $\ell \geqslant 3$. Let χ_{s_1} be the semisimple character of G corresponding to the class of s_1.

In the notation of [3], the conjugacy class of s_1 corresponds to the pair of partitions $\lambda = (1^{\ell-1})$, $\mu = (1^1)$ with $\eta^{(1)} = (\ell-1)$, $\xi^{(1)} = \varnothing$, and $\zeta^{(1)} = (1)$. The semisimple part of the centralizer in G^* is of type $C_1(q) = A_1(q)$ and the p'-part of the order of the centralizer is $(q^2-1)(q^{\ell-1}+1)$. In this case, the centralizer has a unipotent character St of degree q, the Steinberg character. Let χ_1 be the character of G corresponding to the pair $(\chi_{s_1}, \mathrm{St})$. The degree of χ_1 then follows from Lemma 2.1. ∎

Lemma 2.3. *If $G \cong C_\ell(q)$ is of adjoint type with $\ell \geqslant 4$ and q is a power of an odd prime, then G has irreducible characters χ^α, χ_c, and χ_1 with degrees*

$$\begin{aligned}
\chi^\alpha(1) &= q^3 \frac{(q^{2(\ell-2)}-1)(q^{2\ell}-1)}{(q^2-1)^2} \\
\chi_c(1) &= \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^\ell+1} \\
&= (q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1) \\
\chi_1(1) &= q^2 \frac{(q^2+1)(q^6-1)(q^8-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell-2}+1}.
\end{aligned}$$

Proof. In this case, $G \cong \mathrm{PCSp}_{2\ell}(q)$, the projective conformal symplectic group, and the dual group is the spin group $G^* \cong \mathrm{Spin}_{2\ell+1}(q)$, the simply connected group of type B_ℓ. As $C_\ell(q) \cong B_\ell(q)$ when q is even, we will assume q is odd.

The character χ^α is the unipotent character of G corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & \ell-1 \\ & 1 \end{pmatrix}.$$

The degree $\chi^\alpha(1)$ can be computed using the formula in [2, §13.8].

The character χ_c of G is the semisimple character corresponding to the conjugacy class of a regular element s in the Coxeter torus of G^*, as in [11]. Such an element exists by [19, Theorem 1]. In the notation of [3], this conjugacy class corresponds to the pair of partitions $\lambda = (1^\ell)$, $\mu = \varnothing$ with $\eta^{(1)} = (\ell)$ and $\xi^{(1)} = \varnothing$. By [3, §8], the centralizer in G^* is of order $q^\ell + 1$ and the degree $\chi_c(1)$ is as claimed.

The character χ_1 is obtained as follows. Let s_1 be an element of the semisimple conjugacy class of G^* corresponding to the pair of partitions $\lambda = (1^{\ell-2})$, $\mu = (2^1)$ with $\eta^{(1)} = (\ell-2)$, $\xi^{(1)} = \varnothing$, $\zeta^{(2)} = (1)$, and $\omega^{(2)} = \varnothing$, in the notation of [3]. Thus s_1 corresponds to a regular element of the Coxeter torus of $B_{\ell-2}(q)$. Such a regular element exists for all odd q by [19] or [7] since $\ell \geqslant 4$. Let χ_{s_1} be the semisimple character of G corresponding to the class of s_1.

By [3, §8], the semisimple part of the centralizer in G^* is of type $D_2(q)$ and the p'-part of the order of the centralizer is $(q^2-1)^2(q^{\ell-2}+1)$. In this case, the Steinberg character St of the centralizer has degree q^2 and we let χ_1 be the character of G corresponding to the pair $(\chi_{s_1}, \mathrm{St})$. The degree of χ_1 then follows from Lemma 2.1. ∎

LEMMA 2.4. *Let $G \cong D_\ell(q)$ be of adjoint type.*

1. *If $\ell = 4$ and $q > 3$, then G has irreducible characters of degrees*

$$\chi^\beta(1) = \frac{1}{2}q^3\Phi_2^4\Phi_6,\ \chi_c(1) = \Phi_1^4\Phi_2^2\Phi_3\Phi_4^2,\ \text{and}\ \chi_1(1) = q^2\Phi_1^2\Phi_3\Phi_4^2\Phi_6.$$

2. *If $\ell = 5$, then G has an irreducible character of degree*

$$\chi_1(1) = q^2\Phi_1^3\Phi_2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8.$$

3. *If $\ell \geqslant 6$, then G has irreducible characters χ^α, χ_c, and χ_1 with degrees*

$$\begin{aligned}
\chi^\alpha(1) &= q^6\frac{(q^{\ell-4}+1)(q^{2(\ell-3)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q^2-1)^2(q^4-1)}\\
\chi_c(1) &= \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-1}+1)}\\
\chi_1(1) &= q^2\frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-3}+1)}.
\end{aligned}$$

Proof. In this case, $G \cong \mathrm{P}(\mathrm{CO}_{2\ell}(q)^0)$ in the notation of [2, §1.19]. The dual group is the spin group $G^* \cong \mathrm{Spin}_{2\ell}(q)$, the simply connected group of type D_ℓ.

The character χ_c is the semisimple character corresponding to the conjugacy class of a regular element of the Coxeter torus of G^*, as in [11]. Such an element exists for all q when $\ell \geqslant 4$ by [19, Theorem 1]. In the notation of [3], this conjugacy class corresponds to the pair of partitions $\lambda = (1^\ell)$, $\mu = \varnothing$ with $\eta^{(1)} = (\ell-1, 1)$ and $\xi^{(1)} = \varnothing$. By [3, §8], the centralizer in G^* is of order $(q^{\ell-1}+1)(q+1)$ and the degree $\chi_c(1)$ is as claimed in all cases.

The character χ_1 is constructed as follows. Let s_1 be an element of the semisimple conjugacy class of G^* corresponding to the pair of partitions $\lambda = (1^{\ell-2})$, $\mu = (2^1)$ with $\eta^{(1)} = (\ell-3, 1)$, $\xi^{(1)} = \varnothing$, $\zeta^{(2)} = (1)$, and $\omega^{(2)} = \varnothing$, in the notation of [3]. Thus s_1 corresponds to a regular element of the Coxeter torus of $D_{\ell-2}(q)$. For $\ell \geqslant 6$, such a regular element exists for all q by [19] or [7]. Let χ_{s_1} be the semisimple character of G corresponding to the class of s_1.

By [3, §8], the semisimple part of the centralizer in G^* is of type $D_2(q)$ and the p'-part of the order of the centralizer is $(q^2-1)^2(q^{\ell-3}+1)(q+1)$. In this case, the Steinberg character St of the centralizer has degree q^2 and we let χ_1 be the character of G corresponding to the pair $(\chi_{s_1}, \mathrm{St})$. The degree of χ_1 then follows from Lemma 2.1. By [14], the degree $\chi_1(1)$ also exists for $q > 3$ if $\ell = 4$ and for all q if $\ell = 5$.

For $\ell = 4$, the character χ^β is the unipotent character of G corresponding to the symbol

$$\beta = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$$

For $\ell \geqslant 6$, the character χ^α is the unipotent character of G corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & \ell-2 \\ 1 & 2 \end{pmatrix}.$$

The degrees of these unipotent characters can be computed using the formula in [2, §13.8]. ∎

Lemma 2.5. *Let $G \cong {}^2D_\ell(q^2)$ be of adjoint type.*

1. *If $\ell = 4$, then G has irreducible characters of degrees*

$$\begin{aligned} \chi_c(1) &= \Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6, \\ \chi_1(1) &= q^2\Phi_1\Phi_2\Phi_3\Phi_6\Phi_8, \\ \chi_2(1) &= q^3\Phi_1\Phi_3\Phi_4\Phi_8. \end{aligned}$$

2. *If $\ell \geqslant 5$, then G has irreducible characters χ^α, χ_c, and χ_1 with degrees*

$$\begin{aligned} \chi^\alpha(1) &= \frac{1}{2}q^3\frac{(q^{\ell-3}-1)(q^{\ell-2}+1)(q^{\ell-1}-1)(q^\ell+1)}{(q^2+1)(q-1)^2} \\ \chi_c(1) &= (q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1) \\ \chi_1(1) &= q^2\frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell+1)}{q^{\ell-2}+1}. \end{aligned}$$

Proof. In this case, $G \cong \mathrm{P}(\mathrm{CO}_{2\ell}^-(q)^0)$ in the notation of [2, §1.19]. The dual group is the spin group $G^* \cong \mathrm{Spin}_{2\ell}^-(q)$, the simply connected group of type ${}^2D_\ell$.

The character χ^α is the unipotent character of G corresponding to the symbol

$$\alpha = \begin{pmatrix} 0 & 2 & \ell-1 \\ & 1 & \end{pmatrix}.$$

The degree $\chi^\alpha(1)$ can be computed using the formula in [2, §13.8]. (This character also exists when $\ell = 4$, but is not useful in determining the degree graph in that case.)

As in the previous cases, χ_c is the semisimple character corresponding to the conjugacy class of a regular element of the Coxeter torus of G^* used in [11]. Such an element exists for all q when $\ell \geqslant 4$ by [19, Theorem 1]. In the notation of [3], this conjugacy class corresponds to the pair of partitions $\lambda = (1^\ell)$, $\mu = \varnothing$ with $\eta^{(1)} = (\ell)$ and $\xi^{(1)} = \varnothing$. By [3, §8], the centralizer in G^* is of order $q^\ell + 1$ and the degree $\chi_c(1)$ is as claimed in all cases.

The character χ_1 is obtained as follows. Let s_1 be an element of the semisimple conjugacy class of G^* corresponding to the pair of partitions $\lambda = (1^{\ell-2})$, $\mu = (2^1)$ with $\eta^{(1)} = (\ell-2)$, $\xi^{(1)} = \varnothing$, $\zeta^{(2)} = (1)$, and

$\omega^{(2)} = \varnothing$, in the notation of [3]. Thus s_1 corresponds to a regular element of the Coxeter torus of ${}^2D_{\ell-2}(q^2)$. For $\ell \geqslant 6$, such a regular element exists for all q by [19] or [7]. Let χ_{s_1} be the semisimple character of G corresponding to the class of s_1. By [14], the degree $\chi_{s_1}(1)$ also exists for all q if $\ell = 4$ or $\ell = 5$.

By [3, §8], the semisimple part of the centralizer in G^* is of type $D_2(q)$ and the p'-part of the order of the centralizer is $(q^2-1)^2(q^{\ell-2}+1)$. In this case, the Steinberg character St of the centralizer has degree q^2 and we let χ_1 be the character of G corresponding to the pair $(\chi_{s_1}, \mathrm{St})$. The degree of χ_1 then follows from Lemma 2.1. Again the results of [14] show that the character χ_1 also exists for all q if $\ell = 4$ or $\ell = 5$, and the degree is as claimed in these cases as well.

Finally, the character χ_2 is constructed as follows when $\ell = 4$. Let s_2 be an element of the semisimple conjugacy class of G^* corresponding to the pair of partitions $\lambda = \left(1^1, 3^1\right)$, $\mu = \varnothing$ with $\eta^{(1)} = \varnothing$, $\xi^{(1)} = (1)$, $\eta^{(3)} = (1)$, and $\xi^{(3)} = \varnothing$, in the notation of [3]. Let χ_{s_2} be the semisimple character of G corresponding to the class of s_2.

By [3, §8], the semisimple part of the centralizer in G^* is of type ${}^2A_2(q^2)$ and the p'-part of the order of the centralizer is $(q^2-1)^2(q^3+1)$. In this case, the Steinberg character St of the centralizer has degree q^3 and we let χ_2 be the character of G corresponding to the pair $(\chi_{s_2}, \mathrm{St})$. The degree of χ_2 then follows from Lemma 2.1. The results of [14] show that G has $(q-1)^2/2$ characters of this type, so this is a degree of G for all q. ∎

3. PROOF OF THEOREM 1.1

In this section we use the character degrees found in §2 to prove Theorem 1.1. If n is a positive integer, we denote by $\pi(n)$ the set of prime divisors of n. As above, G will denote a finite group $G = \boldsymbol{G}^F$ of adjoint type and L will denote the derived group of G, a finite simple group. Since L is a nonabelian simple group, $\rho(L) = \pi(|L|)$, by the Itô-Michler theorem (see [17, Remarks 13.13]).

The degrees given in §2 are for the group G. If $d = [G : L]$, χ is an irreducible character of G, and μ is an irreducible constituent of the restriction of χ to L, then by [9, Corollary 11.29], $\chi(1)/\mu(1)$ divides d. Moreover, by [2, §12.1], the degrees of the unipotent characters of G and L are the same.

3.1. Groups of Type B_ℓ

Let $\boldsymbol{G}$ be of type B_ℓ, with $\ell \geqslant 2$, so that $G \cong \mathrm{SO}_{2\ell+1}(q)$, $L \cong \Omega_{2\ell+1}(q)$, and $[G : L] = (2, q-1) = d$. The order of L is

$$|L| = \frac{1}{d} q^{\ell^2}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1),$$

and it follows that $\rho(L)$ consists of p and the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell$.

First, let $\ell = 2$ so that $G = B_2(q) \cong C_2(q)$ and $L \cong \mathrm{PSp}_4(q)$. The character table of $\mathrm{Sp}_4(q)$ is computed in [18] for odd q and in [6] for even q.

If $q = 2$, then $L \cong \mathrm{PSp}_4(2)$ is not simple. If $q = 3$, then $|L| = 2^6 \cdot 3^4 \cdot 5$ and, by the Atlas [4] character table, L has the character degree $\chi_{11}(1) = 30 = 2 \cdot 3 \cdot 5$, so $\Delta(L)$ is complete. Hence we may assume $q > 3$.

Since $\ell = 2$, $\rho(L)$ consists of the primes dividing one of q, Φ_1, Φ_2, or Φ_4. The tables in [18] and [6] show that for $q > 3$, L has the character degrees

$$\Phi_1^2\Phi_2^2, \ q\Phi_1\Phi_4, \ \text{and } q\Phi_2\Phi_4.$$

These are the degrees of $\chi_1(j)$, $\chi_7(k)$, and $\chi_9(k)$, respectively, if q is odd, and χ_5, χ_{13}, and χ_{11}, respectively, if q is even. It follows easily that $\Delta(L)$ is a complete graph in this case.

Now let $\ell \geqslant 3$, so that G has the character degrees listed in Lemma 2.2. Since χ^α is a unipotent character, the simple group L also has an irreducible character of degree $\chi^\alpha(1)$.

If q is even, then $d = 1$ and $G = L$. If q is odd, then $d = 2$. Note that since $\ell \geqslant 3$, $\chi_c(1)$ is divisible by $q^2 - 1$ and $\chi_1(1)$ is divisible by $q^{2\ell} - 1$, so also by $q^2 - 1$. Thus both $\chi_c(1)/2$ and $\chi_1(1)/2$ are divisible by 2. Hence the degrees of the irreducible constituents of the restrictions of χ_c and χ_1 to L are divisible by the same primes as χ_c and χ_1.

As noted above,

$$\chi_1(1) = q\frac{(q^4 - 1)(q^6 - 1)\cdots(q^{2(\ell-1)} - 1)(q^{2\ell} - 1)}{q^{\ell-1} + 1}$$

is divisible by $q^2 - 1$ and so by p and all Φ_j, Φ_{2j} for $j = 1, \ldots, \ell$, other than $\Phi_{2(\ell-1)}$. Therefore all primes in $\rho(L)$ other than those dividing only $\Phi_{2(\ell-1)}$ are adjacent in $\Delta(L)$.

The degree

$$\chi_c(1) = (q^2 - 1)(q^4 - 1)\cdots(q^{2(\ell-1)} - 1)(q^\ell - 1)$$

is divisible by $\Phi_{2(\ell-1)}$ and all primes in $\rho(L)$ except p and primes dividing only $\Phi_{2\ell}$. Hence the primes dividing $\Phi_{2(\ell-1)}$ are adjacent in $\Delta(L)$ to all primes except p and primes dividing $\Phi_{2\ell}$.

Finally, since $\ell \geqslant 3$, the unipotent degree

$$\chi^\alpha(1) = \frac{1}{2}q^4\frac{(q^{\ell-2} - 1)(q^{\ell-1} - 1)(q^{\ell-1} + 1)(q^\ell + 1)}{(q^2 - 1)^2}$$

is divisible by $\frac{1}{2}q^4\Phi_{2(\ell-1)}\Phi_{2\ell}$. Hence $\chi^\alpha(1)$ is divisible by p and all primes dividing $\Phi_{2(\ell-1)}$ or $\Phi_{2\ell}$, except possibly 2 (when q is odd). But when q is odd, both $\chi_1(1)$ and $\chi_c(1)$ are divisible by 2, and therefore 2 is adjacent in $\Delta(L)$ to all primes. Hence all primes dividing $\Phi_{2(\ell-1)}$ are adjacent to p and the primes dividing $\Phi_{2\ell}$, and so $\Delta(L)$ is complete.

3.2. Groups of Type C_ℓ

Let $\boldsymbol{G}$ be of type C_ℓ, with $\ell \geqslant 2$, so that $G \cong \mathrm{PCSp}_{2\ell}(q)$, $L \cong \mathrm{PSp}_{2\ell}(q)$, and $[G:L] = (2, q-1) = d$. If q is even, then $C_\ell(q) \cong B_\ell(q)$ and we have shown that the degree graph of a simple group of type B_ℓ is complete. We may therefore assume here that q is odd. Moreover, we have $C_2(q) \cong B_2(q)$ for any q, hence we may assume $\ell \geqslant 3$.

First, let $\ell = 3$ so that $G \cong \mathrm{PCSp}_6(q)$,

$$|G| = q^9(q^2-1)(q^4-1)(q^6-1) = q^9\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6,$$

and $[G:L] = d = 2$. The character table of $\mathrm{CSp}_6(q)$ was computed by Lübeck in [13] and is available in the CHEVIE system [8].

Since $q > 2$, the CHEVIE character table shows that $\mathrm{CSp}_6(q)$ has an irreducible character $\chi = \chi_{82}(1,1,q-2)$ whose kernel contains the center of $\mathrm{CSp}_6(q)$. Hence we may view χ as a character of G. The degree of χ is $\chi(1) = q\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6$, and the degree of a constituent of the restriction χ_L of χ to L is $\chi(1)$ or $\chi(1)/2$. Since q is odd, $\chi(1)$ is divisible by 4, so $\chi(1)$ and $\chi(1)/2$ are divisible by the same primes. It follows that a constituent of χ_L is divisible by every prime in $\rho(L)$ and $\Delta(L)$ is complete in this case.

We may now assume that q is odd and $\ell \geqslant 4$, so that $G \cong \mathrm{PCSp}_{2\ell}(q)$ and $[G:L] = d = 2$. The order of L is

$$|L| = \frac{1}{2}q^{\ell^2}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1),$$

and it follows that $\rho(L)$ consists of p and the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell$.

Since $\ell \geqslant 4$, G has the character degrees listed in Lemma 2.3. As χ^α is unipotent, L also has a unipotent character of degree $\chi^\alpha(1)$. Also, $[G:L] = 2$ implies that irreducible constituents of $(\chi_1)_L$ and $(\chi_c)_L$ have degrees divisible by $\chi_1(1)/2$ and $\chi_c(1)/2$, respectively. We have that $\chi_1(1)$ is divisible by $q^{2\ell}-1$ and $\chi_c(1)$ is divisible by q^2-1, hence both degrees are divisible by 4. It follows that L has irreducible characters whose degrees are divisible by the same primes as $\chi^\alpha(1)$, $\chi_1(1)$, and $\chi_c(1)$.

As noted above,

$$\chi_1(1) = q^2\frac{(q^2+1)(q^6-1)(q^8-1)\cdots(q^{2(\ell-1)}-1)(q^{2\ell}-1)}{q^{\ell-2}+1}$$

is divisible by $q^{2\ell}-1$, hence by Φ_1 and Φ_2. It follows that $\chi_1(1)$ is divisible by p and all Φ_j, Φ_{2j} for $j = 1, \ldots, \ell$, other than $\Phi_{2(\ell-2)}$. Hence all primes in $\rho(L)$ except those dividing only $\Phi_{2(\ell-2)}$ are adjacent in the graph $\Delta(L)$.

The degree

$$\chi_c(1) = (q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1)$$

is divisible by Φ_j for $j = 1, \ldots, \ell$ and by Φ_{2j} for $j = 1, \ldots, \ell - 1$. Hence primes dividing $\Phi_{2(\ell-2)}$ are adjacent to all primes in $\rho(L)$ except p and primes dividing $\Phi_{2\ell}$.

Finally, since $\ell \geqslant 4$, the degree

$$\chi^\alpha(1) = q^3 \frac{(q^{2(\ell-2)} - 1)(q^{2\ell} - 1)}{(q^2 - 1)^2}$$

is divisible by p, $\Phi_{2(\ell-2)}$, and $\Phi_{2\ell}$. Hence primes dividing $\Phi_{2(\ell-2)}$ are adjacent to p and primes dividing $\Phi_{2\ell}$ and it follows that $\Delta(L)$ is a complete graph in this case.

3.3. Groups of Type D_ℓ

Let $\boldsymbol{G}$ be of type D_ℓ with $\ell \geqslant 4$, so that $G \cong \mathrm{P}(\mathrm{CO}_{2\ell}(q)^0)$ (in the notation of [2, §1.19]), $L \cong \mathrm{P}\Omega^+_{2\ell}(q)$, and $[G : L] = (4, q^\ell - 1) = d$.

First, let $\ell = 4$ so that $L \cong \mathrm{P}\Omega^+_8(q)$, or $L \cong \mathrm{O}^+_8(q)$ in Atlas [4] notation. If $q = 2$, then $|L| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ and by the character table of L in [4], L has an irreducible character χ_{11} of degree $\chi_{11}(1) = 210 = 2 \cdot 3 \cdot 5 \cdot 7$. Similarly, if $q = 3$, then $|L| = 2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ and L has the degree $\chi_{17}(1) = 5460 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. Hence $\Delta(L)$ is a complete graph if $q = 2$ or $q = 3$.

We now assume $\ell = 4$ and $q > 3$. In this case,

$$|G| = q^{12}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^4 - 1) = q^{12}\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_6$$

and by Lemma 2.4, G has character degrees

$$\chi_1(1) = q^2\Phi_1^2\Phi_3\Phi_4^2\Phi_6,\ \chi_c(1) = \Phi_1^4\Phi_2^2\Phi_3\Phi_4^2,\ \text{and}\ \chi^\beta(1) = \frac{1}{2}q^3\Phi_2^4\Phi_6.$$

Note that when $d \neq 1$, q is odd and each of these degrees is divisible by 8. Thus, since $[G : L] = (4, q^4 - 1) = 4$ when q is odd, the degrees of the irreducible constituents of the restrictions of these characters to L are divisible by the same primes as the degrees of G.

Now $\chi_1(1)$ is divisible by all primes in $\rho(L)$ except those dividing Φ_2. Each of $\chi_c(1)$ and $\chi^\beta(1)$ is divisible by Φ_2, and every prime in $\rho(L)$ divides at least one of these degrees. Hence $\Delta(L)$ is a complete graph when $\ell = 4$.

We next assume $\ell = 5$. In this case,

$$|G| = q^{20}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)(q^5 - 1) = q^{20}\Phi_1^5\Phi_2^4\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8$$

and by Lemma 2.4, G has character degree $\chi_1(1) = q^2\Phi_1^3\Phi_2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8$. Again, when $d \neq 1$, this degree is divisible by 8 and so L has an irreducible character whose degree is divisible by the same primes as $\chi_1(1)$. All primes in $\rho(L)$ divide this degree, hence $\Delta(L)$ is complete when $\ell = 5$.

We may now assume that $\ell \geqslant 6$. The order of L is

$$|L| = \frac{1}{d} q^{\ell(\ell-1)}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell-1),$$

and it follows that $\rho(L)$ consists of p, the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell-1$, and the primes dividing Φ_ℓ.

Since $\ell \geqslant 6$, G has the character degrees listed in Lemma 2.4, part (3). The character χ^α is unipotent and therefore L also has an irreducible unipotent character of degree $\chi^\alpha(1)$. Moreover, $\chi_1(1)$ and $\chi_c(1)$ are both divisible by $q^{2(\ell-2)}-1$, hence both are divisible by 8 when q is odd. We have $d = 1$ when q is even and $d = 2$ or $d = 4$ when q is odd. Thus, in any case, L has irreducible characters whose degrees are divisible by the same primes as $\chi_1(1)$, $\chi_c(1)$, and $\chi^\alpha(1)$.

The degree

$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-3}+1)}$$

is divisible by all primes in $\rho(L)$ except those dividing only $\Phi_{2(\ell-3)}$. The degree

$$\chi_c(1) = \frac{(q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q+1)(q^{\ell-1}+1)}$$

is divisible by $\Phi_{2(\ell-3)}$ and all other primes in $\rho(L)$ except p and those primes dividing only $\Phi_{2(\ell-1)}$. Therefore it remains to show primes dividing $\Phi_{2(\ell-3)}$ are adjacent to p and primes dividing $\Phi_{2(\ell-1)}$. Finally, since $\ell \geqslant 6$, the degree

$$\chi^\alpha(1) = q^6 \frac{(q^{\ell-4}+1)(q^{2(\ell-3)}-1)(q^{2(\ell-1)}-1)(q^\ell-1)}{(q^2-1)^2(q^4-1)}$$

is divisible by p, $\Phi_{2(\ell-3)}$, and $\Phi_{2(\ell-1)}$. Therefore $\Delta(L)$ is a complete graph in this case as well.

3.4. Groups of Type ${}^2D_\ell$

Let $\boldsymbol{G}$ be of type ${}^2D_\ell$ with $\ell \geqslant 4$, so that $G \cong \mathrm{P}(\mathrm{CO}_{2\ell}^-(q)^0)$ (in the notation of [2, §1.19]), $L \cong \mathrm{P}\Omega_{2\ell}^-(q)$, and $[G : L] = (4, q^\ell+1) = d$.

First, let $\ell = 4$ so that $L \cong \mathrm{P}\Omega_8^-(q)$, or $L \cong \mathrm{O}_8^-(q)$ in Atlas [4] notation. In this case,

$$|G| = q^{12}(q^2-1)(q^4-1)(q^6-1)(q^4+1) = q^{12}\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6\Phi_8$$

and $[G : L] = (4, q^4+1) = d$ is 1 if q is even and 2 if q is odd. By Lemma 2.5, G has character degrees

$$\begin{aligned}
\chi_1(1) &= q^2\Phi_1\Phi_2\Phi_3\Phi_6\Phi_8, \\
\chi_c(1) &= \Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6, \\
\chi_2(1) &= q^3\Phi_1\Phi_3\Phi_4\Phi_8.
\end{aligned}$$

If q is odd, each of these degrees is divisible by 8, hence L has irreducible characters whose degrees are divisible by the same primes as $\chi_1(1)$, $\chi_c(1)$, and $\chi_2(1)$.

The degree $\chi_1(1)$ is divisible by all primes in $\rho(L)$ except those dividing only Φ_4, while $\chi_c(1)$ is divisible by Φ_4 and all other primes except p and the primes dividing Φ_8. It remains only to show that primes dividing Φ_4 are adjacent to p and the primes dividing Φ_8. Since $\chi_2(1)$ is divisible by p, Φ_4, and Φ_8, $\Delta(L)$ is a complete graph in this case.

We now assume $\ell \geqslant 5$. The order of L is

$$|L| = \frac{1}{d} q^{\ell(\ell-1)}(q^2-1)(q^4-1)\cdots(q^{2(\ell-1)}-1)(q^\ell+1),$$

and it follows that $\rho(L)$ consists of p, the primes dividing Φ_j or Φ_{2j} for $j = 1, \ldots, \ell-1$, and the primes dividing $\Phi_{2\ell}$.

Since $\ell \geqslant 5$, G has the character degrees listed in Lemma 2.5, part (2). The character χ^α is unipotent. Hence L also has an irreducible unipotent character of degree $\chi^\alpha(1)$. Moreover, $\chi_1(1)$ and $\chi_c(1)$ are both divisible by $q^{2(\ell-1)}-1$, hence both are divisible by 8 when q is odd. We have $d = 1$ when q is even and $d = 2$ or $d = 4$ when q is odd. Thus, in any case, L has irreducible characters whose degrees are divisible by the same primes as $\chi_1(1)$, $\chi_c(1)$, and $\chi^\alpha(1)$.

The degree

$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)(q^\ell+1)}{q^{\ell-2}+1}$$

is divisible by all primes in $\rho(L)$ except those dividing only $\Phi_{2(\ell-2)}$. The degree

$$\chi_c(1) = (q^2-1)(q^4-1)\cdots(q^{2(\ell-2)}-1)(q^{2(\ell-1)}-1)$$

is divisible by $\Phi_{2(\ell-2)}$ and all other primes in $\rho(L)$ except p and the primes dividing $\Phi_{2\ell}$. Hence it remains only to prove that primes dividing $\Phi_{2(\ell-2)}$ are adjacent to p and the primes dividing $\Phi_{2\ell}$. Finally, since $\ell \geqslant 5$, the degree

$$\chi^\alpha(1) = \frac{1}{2} q^3 \frac{(q^{\ell-3}-1)(q^{\ell-2}+1)(q^{\ell-1}-1)(q^\ell+1)}{(q^2+1)(q-1)^2}$$

is divisible by $\frac{1}{2}q^3\Phi_{2(\ell-2)}\Phi_{2\ell}$. Hence $\chi^\alpha(1)$ is divisible by p and all primes dividing $\Phi_{2(\ell-2)}$ or $\Phi_{2\ell}$, except possibly 2 (when q is odd). But when q is odd, both $\chi_1(1)$ and $\chi_c(1)$ are divisible by 2, and therefore 2 is adjacent in $\Delta(L)$ to all primes. Hence all primes dividing $\Phi_{2(\ell-2)}$ are adjacent to p and the primes dividing $\Phi_{2\ell}$, and $\Delta(L)$ is a complete graph in this case as well.

This completes the proof of Theorem 1.1.

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