Diameters of Degree Graphs of Nonsolvable Groups, II

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ABSTRACT

Let $G$ be a finite group and let $\text{cd}(G)$ be the set of irreducible character degrees of $G$. The degree graph $\Delta(G)$ is the graph whose set of vertices is the set of primes that divide degrees in $\text{cd}(G)$, with an edge between $p$ and $q$ if $pq$ divides $a$ for some degree $a \in \text{cd}(G)$. It is shown using the degree graphs of the finite simple groups that if $G$ is a nonsolvable group, then the diameter of $\Delta(G)$ is at most 3.

1. INTRODUCTION

Throughout this paper, $G$ is a finite group and $\text{Irr}(G)$ is the set of irreducible characters of $G$. We are particularly interested in the values these characters take on the identity of $G$. If $\chi \in \text{Irr}(G)$, then $\chi(1)$ is the degree of $\chi$. The set of all degrees for $G$ is written $\text{cd}(G) = \{\chi(1) | \chi \in \text{Irr}(G)\}$. One tool to study $\text{cd}(G)$ is the graph $\Delta(G)$ whose set of vertices is $\rho(G)$, the set of primes that divide degrees in $\text{cd}(G)$, and there is an edge between $p$ and $q$ if $pq$ divides $a$ for some degree $a \in \text{cd}(G)$.

The distance $d(p, q)$ between two connected vertices $p$ and $q$ in $\Delta(G)$ is the minimum number of edges in a path between the two vertices. The diameter of $\Delta(G)$ is the maximum value of $d(p, q)$ for connected vertices $p, q$. If $\Delta(G)$ is disconnected, then the diameter of $\Delta(G)$ is the largest diameter of a connected component of $\Delta(G)$. The question we consider in this paper is: what are the diameters of graphs that arise in this fashion?

When $G$ is a solvable group, we know that the diameter of $\Delta(G)$ is at most 3. (See [13].) In [11], we showed that if $G$ is a nonsolvable group, then $\Delta(G)$ has diameter at most 4. In that paper, we mentioned that we did not know of any example of a group $G$ where $\Delta(G)$ has diameter 4, and we mentioned that we suspected that the true bound is 3. In this paper, we will prove that the correct bound is indeed 3.

**Main Theorem.** If $G$ is a nonsolvable group, then the diameter of $\Delta(G)$ is at most 3.
Combining this with the result for solvable groups found in [13], we see that if \( G \) is any group, then the diameter of \( \Delta(G) \) is at most 3.

We can find both solvable and nonsolvable groups where \( \Delta(G) \) has diameter 3. Up to abelian factors, the only nonsolvable example that we know of is \( J_1 \), the first sporadic simple Janko group. A solvable example is found in [9]. Carrie Dugan has shown that the example in that paper is the first of a family of examples. This shows that the bound in the Main Theorem cannot be improved any further.

2. OUTLINE OF PROOF

The argument underlying the proof of the Main Theorem is the same as the argument used in [11] to show that the diameter of \( \Delta(G) \) is at most 4. The improvement in our bound comes from better knowledge of the degree graphs of the simple groups. The degree graphs for the alternating groups were determined by Barry and Ward in [1] and the graphs for the groups of Lie type were determined by White in [17], [18], and [19].

The key idea underlying our proof is the following fact, which is proved as Corollary 3.4 of [13].

**Theorem 2.1** (Manz, Willems, Wolf). Let \( M/N \) be a nonsolvable chief factor of a finite group \( G \), and assume that \( G/M \) is abelian. Write \( M/N \) as a direct product of \( t \) copies of a simple group \( S \).

1. If \( t > 1 \), then \( \Delta(G) \) has diameter at most 3.
2. If \( t = 1 \) and \( \Delta(S) \) is connected, then the diameter of \( \Delta(G) \) is at most 2 more than the diameter of \( \Delta(S) \).

Let \( G \) be a nonsolvable finite group. We proved in [11] that if \( \Delta(G) \) is disconnected, then \( \Delta(G) \) has diameter at most 2, so we may assume that \( \Delta(G) \) is connected.

The proof of the Main Theorem will split into two cases. We will first assume \( G' = G'' \). In this case, the hypotheses of Theorem 2.1 will hold. We will show that the diameter of \( \Delta(G) \) is at most 3 when \( t = 1 \) and either \( \Delta(S) \) is disconnected or the diameter of \( \Delta(S) \) is greater than 1. In the second case, we will assume \( G'' < G' \). For this situation we will proceed by induction on \( |G| \), with the first case as the base step, using the techniques found in Chapter 12 of [7].

We conclude this section by mentioning the notation that is used throughout the paper. If \( n \) is an integer, then \( \pi(n) \) is the set of primes that divide \( n \). If \( N \) is a subgroup of \( G \) and \( \theta \in \text{Irr}(N) \), then \( \text{Irr}(G \mid \theta) \) is the set of irreducible constituents of \( \theta^N \). We also write

\[
\text{cd}(G \mid \theta) = \{ \chi(1) \mid \chi \in \text{Irr}(G \mid \theta) \}.
\]
3. THE CASE $G' = G''$

As noted above, in the case where $G' = G''$, we need to prove that the diameter of $\Delta(G)$ is at most 3 when $t = 1$ and $\Delta(S)$ is not complete, in the notation of Theorem 2.1. We now outline the known results regarding the character degree graphs of simple groups with a particular eye toward determining which groups have a degree graph that is not complete. Throughout this discussion, $S$ is a nonabelian simple group.

If $S$ is a sporadic simple group, then $\Delta(S)$ can be computed via [2] and was described in [11]. In all cases, $\Delta(S)$ is connected. In fact, $\Delta(S)$ is a complete graph except when $S$ is $M_{11}$, $M_{23}$, or $J_1$. The graphs of $M_{11}$ and $M_{23}$ have diameter 2 and the graph of $J_1$ has diameter 3.

Suppose $S$ is the alternating group $\text{Alt}(n)$ where $n \geq 5$. For $5 \leq n \leq 14$, the graph $\Delta(S)$ can be determined via the Atlas [2] or GAP [3]. When $n \geq 15$, this graph can be determined using the results in [1]. In particular, $\Delta(S)$ is a complete graph except when $n = 5, 6, or 8$. The graphs when $n = 5$ or 6 are disconnected, and the graph when $n = 8$ is connected of diameter 2. Recall that $\text{Alt}(5) \cong \text{PSL}_2(4) \cong \text{PSL}_2(5)$, $\text{Alt}(6) \cong \text{PSL}_2(9)$, and $\text{Alt}(8) \cong \text{PSL}_4(2)$. For the purposes of this paper, we will treat these groups as groups of Lie type.

If $S$ is a simple group of exceptional Lie type, then $\Delta(S)$ is a complete graph except when $S$ is a Suzuki group $\text{Sz}(2^{2m+1})$ with $m \geq 1$, and the graphs of the Suzuki groups have diameter 2 (see [17]). Finally, we come to the simple groups of classical Lie type. We know that $\Delta(S)$ is disconnected if and only if $S \cong \text{PSL}_2(q)$ for some prime power $q \geq 4$ (see [10, Theorem 2.1]). If $S$ is any other classical simple group, then $\Delta(S)$ is a complete graph except when $S$ is $\text{PSL}_3(q^2)$ and either $q = 4$ or $\pi(q - 1) \not\subseteq \{2, 3\}$, $\text{PSU}_3(q^2)$ and $\pi(q + 1) \not\subseteq \{2, 3\}$, or $\text{PSL}_4(2)$. In each of these cases, $\Delta(S)$ has diameter 2. (This is proved in [18] and [19].)

We now summarize the results for simple groups:

**Theorem 3.1.** [19, Corollary 1.2] If $S$ is a finite simple group, then $\Delta(S)$ is connected unless $S \cong \text{PSL}_2(q)$ for some prime power $q \geq 4$, and $\Delta(S)$ is a complete graph unless $S$ is isomorphic to one of the following groups.

1. $M_{11}$, $M_{23}$, $J_1$.
2. $\text{PSL}_2(q)$ where $q \geq 4$.
3. $\text{PSL}_3(q)$ where $q = 4$ or $\pi(q - 1) \not\subseteq \{2, 3\}$.
4. $\text{PSU}_3(q^2)$ where $\pi(q + 1) \not\subseteq \{2, 3\}$.
5. $\text{Alt}(8) \cong \text{PSL}_4(2)$.
6. $\text{Sz}(2^{2m+1})$ where $m \geq 1$.
Our goal in this section is to prove the following theorem.

**Theorem 3.2.** If $G$ is a finite nonsolvable group such that $G' = G''$ and $\Delta(G)$ is connected, then the diameter of $\Delta(G)$ is at most 3.

**Proof.** Let $M = G'$ and let $N$ be a normal subgroup of $G$ so that $M/N$ is a chief factor for $G$. Since $M/N$ is not abelian, $M/N$ is the direct product of $t$ copies of a nonabelian simple group $S$. If $t > 1$ or if $t = 1$ and $\Delta(S)$ is a complete graph, then the result follows from Theorem 2.1.

If $t = 1$ and $\Delta(S)$ is not a complete graph, then $S$ is isomorphic to one of the groups listed in Theorem 3.1. For each such simple group $S$, we prove that the diameter of $\Delta(G)$ is at most 3 in a series of lemmas.

The case where $S$ is $J_1$ is dealt with in Lemma 3.4. The case where $S$ is $M_{11}$, $M_{23}$, or $\text{Alt}(8)$ is handled via Lemma 3.7. When $S$ is $\text{PSL}_3(4)$, we use Lemma 3.10, and when $S$ is $\text{PSL}_3(q)$ with $\pi(q - 1) \nsubseteq \{2, 3\}$, we use Lemma 3.9. If $S$ is $\text{PSU}_3(q^2)$ with $\pi(q + 1) \nsubseteq \{2, 3\}$, then we apply Lemma 3.11. When $S$ is $\text{Sz}(2^{2m+1})$, the theorem is proved in Lemma 3.12. Finally, we are left with $S$ being $\text{PSL}_2(q)$. If $q > 5$ is odd, then we use Lemma 3.13. If $q$ is even or $q = 5$, then we use Lemma 3.17, where for $q = 5$ we have $\text{PSL}_2(5) \cong \text{PSL}_2(4)$. This exhausts all cases, and so the theorem is proved.

In the lemmas referenced in the proof of Theorem 3.2, we will assume the hypotheses and notation introduced in the proof. With this in mind, we make the following hypothesis.

**Hypothesis 3.3.** $G$ is a finite group so that $\Delta(G)$ is connected, $G'' = G'$, and if $N$ is a normal subgroup of $G$ contained in $M = G'$ such that $M/N$ is a chief factor of $G$, then $M/N$ is isomorphic to a nonabelian simple group $S$.

**3.1. When $\Delta(S)$ is Connected**

We now consider groups satisfying Hypothesis 3.3 for which $\Delta(S)$ is connected. Because the diameter of $\Delta(J_1)$ is 3, it was necessary to consider the case $S \cong J_1$ in [11], where we proved the following result.

**Lemma 3.4.** [11, Lemma 3.5] If $G$ satisfies Hypothesis 3.3 and $S \cong J_1$, then $\Delta(G)$ has diameter at most 3.

The proofs in the cases where $\Delta(S)$ is connected all have the same basic structure. The following two lemmas encode the main arguments underlying all of these cases. The proofs of these lemmas are reminiscent of the proof for $S \cong J_1$ in [11]. We note that these next two lemmas could be applied to $J_1$ with $r = 7$ and $s = 19$. We should also note that if $\Delta(S)$ is connected and $S$ is a nonabelian simple group other than $J_1$, then $\Delta(S)$ has diameter at most 2, and hypothesis (3) of this next lemma is satisfied. We chose to state this lemma in a manner that shows that the $J_1$ case is essentially the same as the cases we consider here.
Lemma 3.5. Suppose that $G$ satisfies Hypothesis 3.3 where $\Delta(S)$ is connected. Assume that there exist primes $r, s \in \rho(S)$ so that the following conditions hold:

1. $rs$ divides some degree $a \in \text{cd}(S)$.
2. Every prime $p \in \rho(G) - \rho(S)$ is adjacent in $\Delta(G)$ to either $r$ or $s$.
3. Every prime $q \in \rho(S)$ has $d(q, r) \leq 2$ and $d(q, s) \leq 2$.

Then $\Delta(G)$ has diameter at most 3.

Proof. Consider $p, q \in \rho(G)$. We know that $\Delta(S)$ has diameter at most 3, so $d(p, q) \leq 3$ if $p, q \in \rho(S)$. Next suppose one of $p$ or $q$ lies in $\rho(G) - \rho(S)$ and the other lies in $\rho(S)$, say $p \in \rho(G) - \rho(S)$ and $q \in \rho(S)$. We know that $d(q, r) \leq 2$ and $d(q, s) \leq 2$. Since $p$ is adjacent to either $r$ or $s$, it follows that $d(p, q) \leq 3$. Finally, suppose that $p, q \in \rho(G) - \rho(S)$. We know that $p$ is adjacent to either $r$ or $s$ and $q$ is adjacent to $r$ or $s$. Since $d(r, s) = 1$, it follows that $d(p, q) \leq 3$, which proves the lemma.

The next lemma is used to show that hypothesis (2) of Lemma 3.5 holds. When the outer automorphism group and the Schur multiplier are trivial, it is immediate that this lemma implies the hypothesis. In the cases where those groups are nontrivial, we have to do additional work. Note that since $S$ is nonabelian simple, the Schur representation group of $S$ is unique.

Lemma 3.6. Suppose that $G$ satisfies Hypothesis 3.3. Assume that there exist primes $r, s \in \rho(S)$ so that for every maximal subgroup $K$ of $S$, either $r$ or $s$ divides $|S : K|$. Let $C/N = C_{G/N}(M/N)$. If $p \in \rho(G) - \rho(S)$, then one of the following occurs:

1. $p$ is adjacent to $r$ or $s$ in $\Delta(G)$.
3. $S$ has a Schur representation group $\Gamma$ with a character $\lambda \in \text{Irr}(Z(\Gamma))$ so that neither $r$ nor $s$ divides the degree of any character in $\text{Irr}(\Gamma | \lambda)$.

Proof. If $p$ divides $|G : CM|$, then we are done, so we suppose that $p$ does not divide $|G : CM|$. Note that $CM$ is a normal subgroup of $G$, so $p \in \rho(CM)$. We know that $\Delta(CM)$ is a subgraph of $\Delta(G)$. If we can show that $p$ is adjacent to $r$ or $s$ in $\Delta(CM)$, the result will hold in $G$, and hence, without loss of generality, we may assume that $G = CM$.

Since $p \notin \rho(S)$, we may use the Ito-Michler theorem to see that $p$ does not divide $|S|$, and thus $p \in \rho(C)$. In particular, there is a character $\theta \in \text{Irr}(C)$ so that $p$ divides $\theta(1)$. Let $T$ be the stabilizer of $\theta$ in $G$.

Suppose $T < G$. Then $T/C$ is contained in some maximal subgroup $H/C$ of $G/C \cong S$. We know that $r$ or $s$ divides $|G : H|$. Since $|G : H|$
divides \(|G : T|\) and \(\theta(1)|G : T|\) divides degrees in \(\text{cd}(G)\), we see that \(p\) is adjacent to either \(r\) or \(s\) as desired.

Now suppose that \(T = G\). Then \((G, C, \theta)\) is character triple isomorphic to \((\Gamma, Z, \lambda)\), where \(\Gamma\) is a representation group for \(S\), \(Z\) is the center of \(\Gamma\), and \(\lambda \in \text{Irr}(Z)\). If \(r\) or \(s\) divides some degree in \(\text{cd}(\Gamma | \lambda)\), then \(pr\) or \(ps\) will divide some degree in \(\text{cd}(G | \theta)\), and \(p\) is adjacent to either \(r\) or \(s\). This proves the lemma.

We now apply these lemmas to obtain the result when \(S\) is \(M_{11}\), \(M_{23}\), or \(\text{Alt}(8)\).

**Lemma 3.7.** If \(G\) satisfies Hypothesis 3.3 and \(S\) is isomorphic to one of \(M_{11}\), \(M_{23}\), or \(\text{Alt}(8)\), then \(\Delta(G)\) has diameter at most 3.

**Proof.** We begin with \(S \cong M_{11}\). We know from [2] that \(S\) has a trivial Schur multiplier and outer automorphism group and that the indices of maximal subgroups are 11, 12, 55, 66, and 165. Thus each index is divisible by 2 or 11. We may apply Lemma 3.6 to see that every prime \(p \in \rho(G) - \rho(S)\) is adjacent to 2 or 11. Also, we have 44 \(\in \text{cd}(S)\) by [2] and the diameter of \(\Delta(S)\) is 2, so we may apply Lemma 3.5 to obtain the conclusion.

Next, we consider \(S \cong M_{23}\). Again, we use [2] to see that \(S\) has a trivial multiplier and outer automorphism group and that the indices of maximal subgroups are 23, 253, 506, 1288, 1771, and 40320. Computing, we determine that each index is divisible by 2 or 23. By Lemma 3.6, every prime \(p \in \rho(G) - \rho(S)\) is adjacent to 2 or 23. We turn to [2] to find 230 \(\in \text{cd}(S)\). We know that the diameter of \(\Delta(S)\) is 2, so we may apply Lemma 3.5 to obtain the conclusion.

Finally, we consider \(S \cong \text{Alt}(8) \cong PSL_4(2)\). Consulting [2], we see that the indices of the maximal subgroups of \(S\) are 8, 15, 28, 35, and 56, and thus each index is divisible by 2 or 5. We also observe that the outer automorphism group of \(S\) has order 2 and \(S\) has a Schur multiplier of order 2. Since \(G/CM\) is isomorphic to a subgroup of \(\text{Out}(S)\) and \(2 \in \rho(S)\), if \(p \in \rho(G) - \rho(S)\), then \(p\) does not divide \(|G : CM|\). Furthermore, if \(\Gamma\) is a Schur representation group for \(S\) and \(\lambda\) is the nonprincipal irreducible character of the center of \(\Gamma\), then 2 divides every degree in \(\text{cd}(\Gamma | \lambda)\). Thus the conclusions (2) and (3) of Lemma 3.6 cannot occur and so conclusion (1) must occur. It follows that every prime \(p \in \rho(G) - \rho(S)\) is adjacent to either 2 or 5. Using [2] once more, we find 20 \(\in \text{cd}(S)\). Since the diameter of \(\Delta(S)\) is 2, we conclude via Lemma 3.5 that \(\Delta(G)\) has diameter at most 3.

We will use the following lemma to determine the indices of relevant maximal subgroups in the cases \(S \cong PSL_3(q)\) and \(S \cong PSU_3(q^2)\).

**Lemma 3.8.** Let \(S\) be a finite simple group of Lie type defined over a field of characteristic \(r\). If \(M\) is a maximal subgroup of \(S\) containing a Sylow \(r\)-subgroup of \(S\), then \(M\) is a maximal parabolic subgroup of \(S\).
Proof. In the notation of [4, Theorem 2.3.4], $S$ is a $BN$-pair with $B = U \times H$, where $H = B \cap N$ and $U = O_4(B)$ is a Sylow $r$-subgroup of $S$. By [4, Theorem 2.6.7], if $M$ is any subgroup of $S$ containing $U$, then $M$ is normalized by $H$ and $MH$ is a parabolic subgroup of $S$. If $M$ is a maximal subgroup of $S$, then $MH$ is either $M$ or $S$. Since $S$ is simple and $M \trianglelefteq MH$, we have $M = MH$ and $M$ is a parabolic subgroup of $S$. \qed

We now consider $PSL_3(q)$ when $q > 4$. We will consider $PSL_3(4)$ separately. Also, $PSL_3(2) \cong PSL_2(7)$, so this case is handled by Lemma 3.13, and $\Delta(PSL_3(3))$ is a complete graph, so this case is covered by Theorem 2.1.

**Lemma 3.9.** If $G$ satisfies Hypothesis 3.3 and $S \cong PSL_3(q)$, where $q > 4$ is a prime power, then $\Delta(G)$ has diameter at most 3.

Proof. Let $K$ be a maximal subgroup of $S$. Let $r$ be the prime divisor of $q$. If $K$ does not contain a full Sylow $r$-subgroup of $S$, then $r$ will divide $|S : K|$. If $K$ does contain a full Sylow $r$-subgroup of $S$, then by Lemma 3.8, $K$ is a maximal parabolic subgroup of $S$. Since a maximal parabolic subgroup of $PSL_3(q)$ has index $q^2 + q + 1$ (see [8, Proposition 4.1.17]), we deduce that $|S : K| = q^2 + q + 1$. Let $s$ be any prime divisor of $q^2 + q + 1$. We have just shown that the index of every maximal subgroup of $S$ is divisible by either $r$ or $s$. From [2], we see that the Schur representation group of $PSL_3(q)$ is $SL_3(q)$ (since $q > 4$), and thus $PSL_3(q)$ has a trivial Schur multiplier unless 3 divides $q - 1$, in which case the Schur multiplier is of order 3.

Suppose $S$ has a nontrivial Schur multiplier. Let $\Gamma = SL_3(q)$ and take $\lambda$ to be a nonprincipal irreducible character of the center of $\Gamma$. We can use the character table for $SL_3(q)$ found in [15] to see that

$$q(q^2 + q + 1) \in \text{cd}(\Gamma | \lambda).$$

Hence conclusion (3) of Lemma 3.6 cannot occur, and if $p \in \rho(G) - \rho(S)$, then either $p$ divides $|G : CM|$ or $p$ is adjacent to $r$ or $s$.

Next, suppose $p \in \rho(G) - \rho(S)$ and $p$ divides $|G : CM|$. Recall that $G/CM$ is isomorphic to a subgroup of $Out(S)$. Consider an element $\sigma \in G - CM$ so that $\sigma$ induces an automorphism of order $p$ of $CM/C \cong S$. Recall that $G/M$ is abelian, so $A = CM(\sigma)$ is a normal subgroup of $G$. Now $2, 3 \in \rho(S)$ and $p \notin \rho(S)$, and so $p \neq 2, 3$. We know that $Out(S)$ consists of diagonal, field, and graph automorphisms. In this case, the diagonal automorphisms (if they exist) will have order 3 and the graph automorphism has order 2. We conclude that $\sigma$ can be viewed as a field automorphism. Thus, $\sigma$ corresponds to an automorphism of order $p$ of the field $F$ with $q$ elements.

Looking at the character table for $PSL_3(q)$ in [15], one can see that the action of $\sigma$ on the irreducible characters of degree $q(q^2 + q + 1)$ corresponds to the action of $\sigma$ on the set

$$D = \{x^3 | x \in F - \{0\}\}.$$
Note that \(|D| = q - 1\) or \((q - 1)/3\). On the other hand, when \(\sigma\) is viewed as an automorphism of \(F\), its fixed field will have order \(r^a\), where
\[
q = r^{ap} = r^{a(p-1)}/3.
\]
Since \(p \geq 5\) and \(r \geq 2\), we see that \(r^{a(p-1)} \geq 16\). This implies \(r^a \leq q/16\), and hence \(|D| > r^a\). Thus not every element of \(D\) will be fixed by \(\sigma\). It follows that there will be a character \(\theta \in \text{Irr}(CM/C)\) of degree \(q(q^2+q+1)\) that is not invariant under the action of \(\sigma\). Hence \(\theta^A \in \text{Irr}(A)\).

We can therefore conclude that in any case every prime \(p \in \rho(G) - \rho(S)\) is adjacent to either \(r\) or \(s\). From [15], we see that
\[
q(q^2 + q + 1) \in \text{cd}(S).
\]
Also, we know that \(\Delta(S)\) has diameter 2, so we may use Lemma 3.5 to see that \(\Delta(G)\) has diameter at most 3.

The characters from [15] used in the previous lemma do not exist when \(q = 4\), so we handle this case separately by consulting [2].

**Lemma 3.10.** If \(G\) satisfies Hypothesis 3.3 and \(S \cong \text{PSL}_3(4)\), then \(\Delta(G)\) has diameter at most 3.

**Proof.** Using [2], we see that the indices for the maximal subgroups of \(S\) are 21, 56, 120, and 280. Notice that each of these is divisible by either 3 or 7. Also, the outer automorphism group of \(S\) has order 12 and \(3 \in \rho(S)\), so if \(p \in \rho(G) - \rho(S)\), then conclusion (2) of Lemma 3.6 cannot occur. Looking at the character table for the representation group \(\Gamma\) of \(S\) in [2], we see that if \(\lambda\) is a nonprincipal irreducible character of the center of \(\Gamma\), then either 3 or 7 divides some degree in \(\text{cd}(\Gamma | \lambda)\). Thus conclusion (3) of Lemma 3.6 cannot occur and therefore every prime \(p \in \rho(G) - \rho(S)\) is adjacent to either 3 or 7. Consulting [2] one more time, we see that \(63 \in \text{cd}(S)\). Since \(\Delta(S)\) has diameter 2, we may apply Lemma 3.5 to see that \(\Delta(G)\) has diameter at most 3.

We now consider the case where \(S \cong \text{PSU}_3(q^2)\) with \(q \geq 3\). Recall that \(\text{PSU}_3(2^2)\) is solvable.

**Lemma 3.11.** If \(G\) satisfies Hypothesis 3.3 and \(S \cong \text{PSU}_3(q^2)\), where \(q \geq 3\) is a prime power, then \(\Delta(G)\) has diameter at most 3.

**Proof.** Let \(K\) be a maximal subgroup of \(S\). Let \(r\) be the prime divisor of \(q\). If \(K\) does not contain a full Sylow \(r\)-subgroup of \(S\), then \(r\) will divide \([S : K]\). If \(K\) does contain a full Sylow \(r\)-subgroup of \(S\), then by Lemma 3.8, \(K\) must be a maximal parabolic subgroup of \(S\). Since a maximal parabolic subgroup of \(S\) has index \(q^3 + 1 = (q+1)(q^2 - q + 1)\) (see
[8, Proposition 4.1.18]), we conclude that $q^2 - q + 1$ divides $|S : K|$. Let $s$ be a prime divisor of $q^2 - q + 1$. It follows that every maximal subgroup has index divisible by either $r$ or $s$. From [2], we know that the Schur representation group for $S$ will be $SU_3(q^2)$ and the Schur multiplier of $S$ is trivial unless 3 divides $q + 1$, in which case its order is 3.

Suppose $S$ has a nontrivial Schur multiplier and let $\Gamma = SU_3(q^2)$. Take $\lambda$ to be a nonprincipal irreducible character of the center of $\Gamma$. Using the character table for $SU_3(q^2)$ in [15], we see that

$$q(q^2 - q + 1) \in \text{Irr}(\Gamma | \lambda).$$

Hence conclusion (3) of Lemma 3.6 cannot occur, and if $p \in \rho(G) - \rho(S)$, then either $p$ divides $|G : CM|$ or $p$ is adjacent to $r$ or $s$.

Next, suppose $p \in \rho(G) - \rho(S)$ and $p$ divides $|G : CM|$. Recall that $G/\text{CM}$ is isomorphic to a subgroup of $\text{Out}(S)$. Consider an element $\sigma \in G - \text{CM}$ so that $\sigma$ induces an automorphism of order $p$ of $\text{CM}/C \cong S$. Since $G/\text{CM}$ is abelian, $A = \text{CM}(\sigma)$ is a normal subgroup of $G$. We have $2, 3 \in \rho(S)$ and $p \notin \rho(S)$, so $p \neq 2, 3$. We know that $\text{Out}(S)$ consists of diagonal and field automorphisms. In this case, the diagonal automorphisms (if they exist) will have order 3. We conclude that $\sigma$ can be viewed as a field automorphism. Thus $\sigma$ corresponds to an automorphism of order $p$ of the field $F$ with $q^2$ elements.

Looking at the character table for $\text{PSU}_3(q^2)$ in [15], one can see that the action of $\sigma$ on the irreducible characters of degree $q(q^2 - q + 1)$ corresponds to the action of $\sigma$ on the set

$$D = \{x^3|x \in F, x^{q^3 + 1} = 1\}.$$ 

We know that $D$ is a cyclic group and we let $d$ be a generator of $D$. Thus $d$ has order $(q + 1)/e$, where $e = (3, q + 1)$. On the other hand, when $\sigma$ is viewed as an automorphism of $F$, its fixed field $E$ is the subfield of order $r^{2a}$, where $q = r^{ap}$. If $d$ is in $E$, then we would have that $(q + 1)/e = (r^{ap} + 1)/e$ divides $r^{2a} - 1$. This is absurd since

$$r^{ap} + 1 = r^{a(p-2)}r^{2a} + 1 \geq 8r^{2a} + 1 > 3(r^{2a} - 1) \geq e(r^{2a} - 1).$$

Thus, $d$ is not fixed by $\sigma$. It follows that there will be a character $\theta \in \text{Irr}(\text{CM}/C)$ of degree $q(q^2 - q + 1)$ that is not invariant under the action of $\sigma$. Hence $\theta^A \in \text{Irr}(A)$. We conclude that $p$ is adjacent to both $r$ and $s$ in $\Delta(A)$. Since $A$ is a normal subgroup of $G$, it follows that $p$ is adjacent to $r$ and $s$ in $\Delta(G)$.

We can therefore conclude that, in any case, every prime $p \in \rho(G) - \rho(S)$ is adjacent to either $r$ or $s$. From [15], we see that

$$q(q^2 - q + 1) \in \text{cd}(S).$$

Also, we know that $\Delta(S)$ has diameter 2, so we may use Lemma 3.5 to see that $\Delta(G)$ has diameter at most 3.  


The last case from Theorem 3.1 in which $\Delta(S)$ is connected is the case where $S$ is a Suzuki group, which we consider now.

**Lemma 3.12.** *If $G$ satisfies Hypothesis 3.3 and $S \cong Sz(2^{2m+1})$, where $m \geq 1$, then $\Delta(G)$ has diameter at most 3.*

*Proof.* Let $a = 2m + 1$, $q = 2^{2m+1} = 2^a$, and $d = 2^m$, so the order of $S$ is

$$|S| = q^2(q - 1)(q^2 + 1) = 2^{2a}(2^a - 1)(2^{2a} + 1)$$

and

$$q^2 + 1 = (q + 2d + 1)(q - 2d + 1).$$

The subgroups of $S$ are described in [16]. If $K$ is a maximal subgroup of the form $Sz(2^b)$, then $b | a$ and

$$|S : K| = 2^{2(a-b)} \cdot \frac{2^a - 1}{2^b - 1} \cdot \frac{2^{2a} + 1}{2^{2b} + 1}.$$ 

Each maximal subgroup that is not of the form $Sz(2^b)$ has index divisible by either $q + 2d + 1$ or $q - 2d + 1$.

Let $r$ be a Zsigmondy prime for $2^{2a} - 1 = (2^a - 1)(2^{2a} + 1)$; that is, $r$ is a prime that divides $2^{2a} - 1$ but does not divide $2^k - 1$ for any $k < 4a$. (Such a prime exists by Zsigmondy’s theorem. See [14, Theorem 3], for example).

In particular, $r$ does not divide $2^{2a} - 1$, so must divide $2^{2a} + 1$. Also, for $b < a$, $r$ does not divide $2^b - 1$ or $2^{4b} - 1$, hence does not divide $2^{2b} + 1$. Hence $r$ divides $|S : Sz(2^b)|$ for any proper divisor $b$ of $a$. Moreover, $r$ will divide exactly one of $q + 2d + 1$ or $q - 2d + 1$, and we let $s$ be any prime divisor of the other. It follows that the index of each maximal subgroup of $S$ will be divisible by $r$ or $s$.

If the Schur multiplier of $S$ is nontrivial, then $m = 1$, $q + 2d + 1 = 13$, and $q - 2d + 1 = 5$. In this case, we have $r = 13$ and $s = 5$. Suppose $\Gamma$ is the Schur representation group for $S$ and $\lambda$ is a nonprincipal irreducible character of the center of $\Gamma$. Using [2], we see that both $r = 13$ and $s = 5$ divide degrees in $\text{cd}(\Gamma | \lambda)$. Hence conclusion (3) of Lemma 3.6 cannot occur, and if $p \in \rho(G) - \rho(S)$, then either $p$ divides $|G : CM|$ or $p$ is adjacent to $r$ or $s$.

Next, suppose $p \in \rho(G) - \rho(S)$ and $p$ divides $|G : CM|$. Recall that $G/CM$ is isomorphic to a subgroup of $\text{Out}(S)$. By [16, Theorem 11], we know that $\text{Out}(S)$ corresponds to the Galois group of the field $F$ whose order is $q = 2^{2m+1}$, and so is cyclic of order $2m + 1$. Let $\sigma \in G - CM$ correspond to an automorphism of order $p$ of $CM/C$. Now $A = CM(\sigma)$ is a normal subgroup of $G$. If we can show that $p$ is adjacent to $r$ or $s$ in $\Delta(A)$, it will follow that $p$ is adjacent to $r$ or $s$ in $\Delta(G)$. Without loss of generality, we may assume that $G = A$. By the proof of [16, Theorem 11], $\sigma$ will normalize a subgroup $T/C$ of $CM/C$ whose order is $q - 1$, and the action of $\sigma$ on $T/C$ will correspond to the action on the multiplicative
group \( F - \{0\} \). Also, each conjugacy class of elements of order \( q - 1 \) in \( CM/C \) will intersect \( T/C \) in a pair consisting of an element and its inverse. Since \( p \) must be odd, it follows that \( \sigma \) will stabilize the pair \( \{tC, t^{-1}C\} \) if and only if \( \sigma \) stabilizes \( tC \). We see that \( CM/C \) will have conjugacy classes of elements of order \( q - 1 \) that are not stabilized by \( \sigma \). Looking at the character table for \( S \) in [16], we see that the characters in \( \text{Irr}(CM/C) \) of degree \( q^2 + 1 \) are distinguished by their values on the conjugacy classes of elements of order \( q - 1 \). It is not difficult to show that there will be a character \( \theta \in \text{Irr}(CM/C) \) with \( \theta(1) = q^2 + 1 \) that is not stabilized by \( \sigma \).

Hence \( \theta \) is not adjacent to both \( r \) and \( s \). We prove the following result.

**Lemma 3.13.** [11, Lemma 3.7] If \( G \) satisfies Hypothesis 3.3 and \( S \cong \text{PSL}_2(2^n) \), where \( r \) is an odd prime with \( r^n > 2 \), then \( \Delta(G) \) has diameter at most 3.

All that remains is to consider the case where \( S \cong \text{PSL}_2(2^n) \) with \( n \geq 2 \). The case where \( r = 2 \) was also considered in [11]. We proved there that if \( S \cong \text{PSL}_2(2^n) \) with \( n \geq 2 \), then \( \Delta(G) \) has diameter at most 4. The key idea for our proof was the following fact, which we state here in the case \( r = 2 \).

**Lemma 3.14.** [11, Proposition 3.6] Let \( G \) satisfy Hypothesis 3.3, and let \( S \cong \text{PSL}_2(2^n) \) with \( n \geq 2 \). If \( p \in \rho(G) - \rho(S) \), then \( p \) is adjacent to all of the primes in two of the sets \( \{2\} \), \( \pi(2^n - 1) \), and \( \pi(2^n + 1) \).

We note that in the situation of Lemma 3.14, if \( p, q \in \rho(G) - \rho(S) \), then \( p \) and \( q \) must have a common neighbor. Hence \( d(p, q) \leq 2 \).

Our proof here will incorporate most of the ideas of the proof of [11, Proposition 3.6], but we will need to get more information in the current case. We will also use several results from [10]. In particular, we will use the following result. Unfortunately, the proof in [10] is not correct, but a correct proof can be found in [12].
Lemma 3.14. Let \( n \geq 2 \) be an integer and let \( \pi \) be either the set of primes dividing \( 2^n - 1 \) or the set of primes dividing \( 2^n + 1 \). Suppose that \( G \cong \operatorname{SL}_2(2^n) \) acts via automorphisms on a nontrivial group \( V \). Then there is a nonidentity element \( x \in V \) so that \( C_G(x) \) does not contain a Hall \( \pi \)-subgroup of \( G \) as a normal subgroup.

The next lemma extends many of the arguments found in the proof of Lemma 3.14.

Lemma 3.16. Suppose \( G \) satisfies Hypothesis 3.3 and \( S \cong \operatorname{PSL}_2(2^n) \) with \( n \geq 2 \). If there exists a character \( \theta \in \operatorname{Irr}(N) \) so that the stabilizer \( T \) of \( \theta \) in \( M \) satisfies \( T < M \), then the following occur:

1. One of \( 2(2^n - 1) \), \( 2(2^n + 1) \), or \( (2^n - 1)(2^n + 1) \) divides some degree in \( \text{cd}(M \mid \theta) \).

2. If no prime in \( \pi(2^n + 1) \) is adjacent to any prime in \( \{2\} \cup \pi(2^n - 1) \), then \( T/N \) contains a unique Hall \( \pi(2^n + 1) \)-subgroup of \( M/N \).

3. If no prime in \( \pi(2^n - 1) \) is adjacent to any prime in \( \{2\} \cup \pi(2^n + 1) \), then \( T/N \) contains a Hall \( \pi(2^n - 1) \)-subgroup of \( M/N \). If this subgroup is not unique in \( T/N \), then \( n = 2 \), \( T/N \cong \operatorname{Alt}(4) \), and \( \theta \) is fully-ramified with respect to \( O_2(T/N) \).

Proof. To prove this lemma, we will use Dickson’s list of the subgroups of \( \operatorname{PSL}_2(2^n) \), which can be found as Hauptsatz II.8.27 of [5]. The first possibility is that \( T/N \) is an elementary abelian 2-group. In this case, \( (2^n - 1)(2^n + 1) \) divides \( |M : T| \), and so that product divides all degrees in \( \text{cd}(M \mid \theta) \). Obviously, this case cannot occur for conclusion (2) or (3).

The second possibility is that \( T/N \) is cyclic of order \( z \) or dihedral of order \( 2z \), where \( z \) divides \( 2^n - 1 \) or \( 2^n + 1 \). It is easy to see that 2 divides \( |M : T| \). Also, \( 2^n + 1 \) will divide \( |M : T| \) when \( z \) divides \( 2^n - 1 \), and \( 2^n - 1 \) will divide \( |M : T| \) when \( z \) divides \( 2^n + 1 \). Thus either \( 2(2^n + 1) \) or \( 2(2^n - 1) \) divides every degree in \( \text{cd}(M \mid \theta) \). Notice that if (2) occurs, then \( T/N \) will be cyclic of order \( 2^n + 1 \) or dihedral of order \( 2(2^n + 1) \), and \( T/N \) contains a Hall \( \pi(2^n + 1) \)-subgroup of \( M/N \). On the other hand, if (3) occurs, then \( T/N \) will be cyclic of order \( 2^n - 1 \) or dihedral of order \( 2(2^n - 1) \), and \( T/N \) contains a unique Hall \( \pi(2^n - 1) \)-subgroup of \( M/N \).

The third possibility is that \( T/N \) is the semi-direct product of a cyclic group of order \( t \) acting on an elementary abelian 2-group of order \( 2^n \), where \( t > 1 \) divides both \( 2^n - 1 \) and \( 2^n + 1 \) and \( 0 < m \leq n \). In fact, \( T/N \) is a Frobenius group in this case. (Notice that this includes the case where \( T/N \cong \operatorname{Alt}(4) \).) Observe that \( 2^n + 1 \) divides \( |M : T| \). If \( m < n \), then 2 divides \( |M : T| \), and (1) will occur. Also, \( t \leq 2^n - 1 < 2^n - 1 \) implies that some prime from \( \pi(2^n - 1) \) divides \( |M : T| \), and hence this case cannot occur for (2) or (3).
Now, suppose that \( m = n \). Since the Sylow subgroups of \( T/N \) are cyclic for odd primes, we deduce that if \( \theta \) extends to \( K \), where \( K/N = O_2(T/N) \), then \( \theta \) will extend to \( T \) (see [7, Corollary 11.31]). Note that \( K/N \) is now a Sylow 2-subgroup of \( M/N \). We know that \( [T : K] \in \text{cd}(T/N) \), and hence

\[
(2^n - 1)(2^n + 1) = |M : K| = |M : T|[T : K]
\]
divides some degree in \( \text{cd}(M \mid \theta) \), as needed for (1). Notice that this cannot occur for (2) or (3).

On the other hand, if \( \theta \) does not extend to \( K \), then 2 divides every degree in \( \text{cd}(K \mid \theta) \), and so 2 divides every degree in \( \text{cd}(T \mid \theta) \). Using Clifford’s theorem, we conclude that \( 2|M : T| \) divides every degree in \( \text{cd}(M \mid \theta) \), and since \( 2^n + 1 \) divides \( |M : T| \), we have (1) in this case. Also, we see that this case cannot occur for (2). If the hypothesis of (3) holds, then it must be that \( t = 2^n - 1 \), since otherwise we would have a prime dividing \( 2^n - 1 \) that is adjacent to 2 and all the primes in \( \pi(2^n + 1) \). It follows that \( T/N \) contains a Hall \( \pi(2^n - 1) \)-subgroup of \( M/N \).

Since \( n \) is the smallest integer \( k \) so that \( 2^n - 1 \) divides \( 2^k - 1 \), we conclude that \( K/N \) is a chief factor of \( T/N \). We now use [7, Problem 6.12] to see that \( \theta \) is fully-ramified with respect to \( K/N \). We now know that \( n = 2e \) for some integer \( e \). By [6, Theorem 5.7], \( |T : K| = 2^n - 1 \) must divide either \( 2^e - 1 \) or \( 2^e + 1 \). The only way

\[
2^n - 1 = 2^{2e} - 1 = (2^e - 1)(2^e + 1)
\]
can divide \( 2^e - 1 \) or \( 2^e + 1 \) is if \( 2^e - 1 = 1 \) and hence \( e = 1 \). This implies \( n = 2 \), and \( T/N \) is isomorphic to \( \text{Alt}(4) \). This proves the conclusion of (3) in this case.

The final possibility is that \( T/N \) is isomorphic to \( \text{SL}_2(2^m) \), where \( 1 \leq m < n \) and \( m \) divides \( n \). (Note that this includes the case where \( T/N \cong \text{Alt}(5) \) when \( 2^{2n} - 1 \equiv 0 \pmod{5} \), since \( \text{Alt}(5) \cong \text{PSL}_2(4) \) and \( 2^{2n} - 1 \equiv 0 \pmod{5} \) if and only if 2 divides \( n \).) We see that

\[
|M : N| = (2^n - 1)2^n(2^n + 1)
\]
and

\[
|T : N| = (2^n - 1)2^n(2^n + 1).
\]

Since \( m < n \), we conclude that 2 divides \( |M : T| \). It is well-known that \( 2^m - 1 \) will divide \( 2^n - 1 \). If \( n/m \) is even, then it is not difficult to show that \( 2^m + 1 \) also divides \( 2^n - 1 \), and so \( 2^n + 1 \) divides \( |M : T| \), proving (1) in this case. We see that this case cannot occur in (2). If \( n/m \) is odd, then \( 2^m + 1 \) divides \( 2^n + 1 \). This implies that primes in both \( \pi(2^n - 1) \) and \( \pi(2^n + 1) \) divide \( |M : T| \), so this case cannot occur in either (2) or (3). We now work to show that the case where \( n/m \) is even cannot occur in (3) and that conclusion (1) holds when \( n/m \) is odd.
Suppose first that \( m > 2 \). We know the Schur multiplier of \( T/N \) is trivial, so \( \theta \) extends to \( T \). By Gallagher’s theorem, we obtain

\[
(2^m - 1)\theta(1) \in \text{cd}(T \mid \theta),
\]

and hence

\[
|M : T|(2^m - 1)\theta(1) \in \text{cd}(M \mid \theta).
\]

Now suppose that \( m = 2 \). In this case, \( 2^m - 1 = 3 \). If \( \theta \) extends to \( T \), we obtain

\[
3\theta(1) \in \text{cd}(T \mid \theta).
\]

If \( \theta \) does not extend, we can use [2] to see that

\[
6\theta(1) \in \text{cd}(T \mid \theta).
\]

Thus, when \( m = 2 \), we see that \( |M : T|3\theta(1) \) will divide some degree in \( \text{cd}(M \mid \theta) \). Recall that 2 divides \( |M : T| \). In any case, when \( n/m \) is even, the primes in

\[
\pi(2^m - 1) \subseteq \pi(2^n - 1)
\]

are adjacent to 2, and so this case cannot occur for (3). When \( n/m \) is odd, we see that \( 2^n - 1 \) divides \( |M : T|(2^m - 1) \), and this yields conclusion (1).

**Lemma 3.17.** If \( G \) satisfies Hypothesis 3.3 and \( S \cong \text{PSL}_2(2^n) \) with \( n \geq 2 \), then \( \Delta(G) \) has diameter at most 3.

**Proof.** We begin by noting that

\[
\rho(S) = \{2\} \cup \pi(2^n - 1) \cup \pi(2^n + 1).
\]

If \( p, q \in \rho(G) - \rho(S) \), then Lemma 3.14 shows that \( p \) and \( q \) must have a common neighbor, and so the distance between \( p \) and \( q \) is at most 2. Thus we will only need to consider pairs of primes with at most one prime in \( \rho(G) - \rho(S) \).

Suppose that \( \text{Irr}(N) \) contains a nonlinear character \( \theta \) that extends to \( M \). By Gallagher’s theorem, we have \( \theta(1)2^n, \theta(1)(2^n - 1), \) and \( \theta(1)(2^n + 1) \) are all in \( \text{cd}(M) \). Let \( p \) be a prime divisor of \( \theta(1) \). This shows that \( p \) is adjacent in \( \Delta(M) \) to every prime in \( \rho(S) \). Since \( M \) is a normal subgroup of \( G \), \( \Delta(M) \) is a subgraph of \( \Delta(G) \). Thus any two primes in \( \rho(S) \) will have \( p \) as a common neighbor in \( \Delta(G) \), and the distance between primes in \( \rho(S) \) is at most 2. If \( q \) is any prime in \( \rho(G) - \rho(S) \) and \( r \) is a prime in \( \rho(S) \), then we know that \( q \) is adjacent to some prime \( q' \in \rho(S) \), and then

\[
q - q' - p - r
\]

is a path in \( \Delta(G) \). Hence the distance between \( q \) and \( r \) is at most 3. This implies \( \Delta(G) \) has diameter at most 3.
Therefore, we may assume that no nonlinear irreducible character of $N$ extends to $M$. Since $M = M'$, it follows that no nonprincipal linear character of $N$ extends to $M$, and therefore the principal character of $N$ is the only irreducible character of $N$ that extends to $M$.

Let $C/N = C_{G/N}(M/N)$. We know that

$$S \cong CM/C \subseteq G/C \subseteq \text{Aut}(S).$$

We suppose first that $\Delta(G/C)$ is connected. By [10, Theorem 2.7], we see that 2 must divide $|G : CM|$. Also, we see that 2 is adjacent to all of the primes in $\pi(2^n - 1)$ and to all of the primes in $\pi(2^n + 1)$. This implies that the subgraph of $\Delta(G)$ induced by $\rho(S)$ has diameter at most 2. Finally, if $p \in \rho(G) - \rho(S)$, then $p$ is adjacent to all primes in two of the sets $\{2\}$, $\pi(2^n - 1)$, and $\pi(2^n + 1)$ via Lemma 3.14. We know 2 is adjacent to all of the primes in $\pi(2^n - 1) \cup \pi(2^n + 1)$. If $p$ is adjacent to 2, then $p$ will have a distance 2 to all of the primes in $\pi(2^n - 1) \cup \pi(2^n + 1)$. On the other hand, if $p$ is not adjacent to 2, then $p$ will be adjacent to all of the primes in $\pi(2^n - 1) \cup \pi(2^n + 1)$, and $p$ will have a distance 2 to 2. It follows that in this case, $\Delta(G)$ has diameter at most 2.

Thus, we may assume that $\Delta(G/C)$ is not connected. We claim that in fact $\Delta(G/N)$ is disconnected. To see this, first note that since $M = G'$, it follows that

$$[C, G] \subseteq M \cap C = N,$$

and so $C/N$ is central in $G/N$. We may then apply [10, Theorem 6.3] to see that $\Delta(G/N)$ is disconnected. As $\Delta(G)$ is connected, we conclude that $N > 1$.

To complete this proof we consider separately the cases where $n > 2$ and $n = 2$. We first assume that $n > 2$. We know that the Schur multiplier of $S$ is trivial, and as no nonprincipal irreducible character of $N$ extends to $M$, we conclude that the principal character is the only $M$-invariant character in $\text{Irr}(N)$. Since $N > 1$, we can apply Lemma 3.16 with $\theta$ any nonprincipal irreducible character of $N$, to see that at least one of $(2^n - 1)(2^n + 1)$, $2(2^n - 1)$, or $2(2^n + 1)$ divides some degree in $\text{cd}(M | \theta)$.

Suppose first that $(2^n - 1)(2^n + 1)$ divides some degree in $\text{cd}(M)$ and hence in $\text{cd}(G)$. It follows that $\pi(2^n - 1) \cup \pi(2^n + 1)$ induces a complete subgraph of $\Delta(G)$. Since $\Delta(G)$ is connected, 2 is adjacent to some prime $r \in \rho(G)$. If $r$ is not in $\pi(2^n - 1) \cup \pi(2^n + 1)$, then we use Lemma 3.14 to see that $r$ is adjacent to some prime $r' \in \pi(2^n - 1) \cup \pi(2^n + 1)$.

In any case, the distance from 2 to any prime in $\pi(2^n - 1) \cup \pi(2^n + 1)$ will be at most 3. Let $p$ be any prime in $\rho(G) - \rho(S)$. By Lemma 3.14, $p$ will be adjacent to some prime in $\pi(2^n - 1) \cup \pi(2^n + 1)$, and so the distance from $p$ to any prime in $\pi(2^n - 1) \cup \pi(2^n + 1)$ is at most 2. If $p$ is not adjacent
to 2, then in light of Lemma 3.14, \( p \) will be adjacent to all the primes in \( \pi(2^n - 1) \cup \pi(2^n + 1) \). In particular, \( p \) is either adjacent to \( r \) or to \( r' \), and the distance from \( p \) to 2 is at most 3. We see that the diameter of \( \Delta(G) \) is at most 3.

We now suppose that either \( 2(2^n - 1) \) or \( 2(2^n + 1) \) divides some degree in \( cd(M) \) and hence in \( cd(G) \). It follows that either \( \{2\} \cup \pi(2^n - 1) \) or \( \{2\} \cup \pi(2^n + 1) \) induces a complete subgraph of \( \Delta(G) \). Let \( \pi \) be this set of primes, and let \( \rho \) be the set of primes so that

\[
\pi \cup \rho = \{2\} \cup \pi(2^n - 1) \cup \pi(2^n + 1) = \rho(S).
\]

If some prime in \( \rho \) is adjacent to some prime in \( \pi \), then \( \rho(S) \) will induce a subgraph of \( \Delta(G) \) whose diameter is at most 3. Furthermore, if \( p \) is any prime in \( \rho(G) - \rho(S) \), then using Lemma 3.14 we have two possibilities. Either \( p \) is adjacent to all the primes in \( \pi \) or \( p \) is adjacent to some primes in \( \pi \) and all of the primes in \( \rho \). If \( p \) is adjacent to all the primes in \( \pi \), it will follow that the distance from \( p \) to any of the primes in \( \rho \) is at most 3. If \( p \) is adjacent to some primes in \( \pi \) and all the primes in \( \rho \), then the distance from \( p \) to any prime in \( \pi \) is at most 2. This proves that the diameter of \( \Delta(G) \) is at most 3 in this case.

Thus, we may assume that no prime in \( \rho \) is adjacent to any prime in \( \pi \). We show that this leads to a contradiction. We now use (2) and (3) of Lemma 3.16 and the fact that \( n > 2 \) to see that if \( T \) is the stabilizer of any nonprincipal character in \( \text{Irr}(N) \), then \( T/N \) contains a unique Hall \( \rho \)-subgroup of \( M/N \). If \( N' < N \), then we have \( S \cong \text{PSL}_2(q) \) acting on the group \( \text{Irr}(N/N') \), where every nonidentity element of \( \text{Irr}(N/N') \) has a stabilizer with a unique Hall \( \rho \)-subgroup of \( S \), violating Lemma 3.15. The other possibility is \( N' = N \). Let \( L \) be a normal subgroup of \( G \) so that \( N/L \) is a chief factor for \( G \). We know that

\[
N/L = R_1/L \times \cdots \times R_r/L,
\]

where there is a nonabelian simple group \( R \) so that \( R_i/L \cong R \) for \( 1 \leq i \leq r \). Furthermore, we know that \( G \) acts transitively on the set

\[
\{R_i \mid i = 1, \ldots, r\}.
\]

If \( r = 1 \), then \( N/L \) is simple. Let \( B/L = C_{G/L}(N/L) \). We know that \( G/BN \) is isomorphic to a subgroup of the outer automorphism group of \( R \), and thus \( G/BN \) is solvable. Hence \( M \subsetneq BN \). This implies that every irreducible character of \( N/L \) is invariant in \( M \), and we are assuming that the principal character is the only character in \( \text{Irr}(N) \) that is \( M \)-invariant. We must have \( r \geq 2 \). Pick a nonprincipal character \( \phi \in \text{Irr}(R) \), and take \( T \) to be the stabilizer in \( M \) of

\[
\phi \times 1 \times \cdots \times 1 \in \text{Irr}(N/L).
\]

We can find \( g \in G \) so that \( (T \cap T^g)/N \) does not contain a Hall \( \rho \)-subgroup of \( M/N \). Renumbering if necessary, set \( R_2 = R_1^g \). Let \( Y \) be the stabilizer in \( G \) of
\[
\phi \times \phi \times 1 \times \cdots \times 1 \in \text{Irr}(N/L).
\]
It is not difficult to see that \( T \cap T^g \) has index at most 2 in \( Y \), and so \( Y/N \) does not contain a Hall \( \rho \)-subgroup of \( M/N \), which is a contradiction.

To complete the proof of the lemma, we must consider the case where \( n = 2 \). This yields \( \pi(2^n - 1) = \{3\} \) and \( \pi(2^n + 1) = \{5\} \). We now consider \( T \cap T^g \) to see that 3 is adjacent to 5, 2 is adjacent to 3, or 2 is adjacent to 5 in \( \Delta(M) \). In any case, two of the three primes in \( \{2, 3, 5\} \) are adjacent in \( \Delta(M) \), and hence in \( \Delta(G) \). Let \( x \) and \( y \) be the two primes that are adjacent and let \( z \) be the third prime. Since \( \Delta(G) \) is connected, there is a prime \( r \in \rho(G) \) that is adjacent to \( z \). If \( r \) is one of \( x \) or \( y \), then the distance from \( z \) to either \( x \) or \( y \) is at most 2. If \( r \) is not \( x \) or \( y \), then \( r \) is adjacent to one of them by Lemma 3.14. It follows that the distance between \( z \) and \( x \) or \( y \) is at most 3. This implies that the diameter of \( \Delta(G) \) is at most 3.

4. THE CASE \( G'' < G' \)

In this section, we prove the Main Theorem. The proof uses induction on the order of \( G \), with the case where \( G'' = G' \) serving as the base case.

Proof of Main Theorem. We proceed by induction on \( |G| \). If \( G'' = G' \), then the theorem follows by Theorem 3.2. We may therefore now assume that \( G'' < G' \). In other words, we assume \( G \) has a nonabelian solvable quotient.

Take \( N \) to be maximal so that \( N \) is normal in \( G \) and \( G/N \) is solvable and nonabelian. By Lemma 12.3 of [7], we know that either \( G/N \) is a \( p \)-group for some prime \( p \) or \( G/N \) is a Frobenius group with an abelian Frobenius complement.

Suppose first that \( G/N \) is a \( p \)-group for some prime \( p \). Let \( q \) be any prime in \( \rho(G) \), and take \( \chi \in \text{Irr}(G) \) so that \( q \) divides \( \chi(1) \). If \( p \) divides \( \chi(1) \), then \( q \) and \( p \) are adjacent in \( \Delta(G) \). If \( p \) does not divide \( \chi(1) \), then \( \chi_N \) is irreducible, and we can use Gallagher’s theorem to see that \( p^a \chi(1) \in \text{cd}(G) \), where \( p^a \) is a nontrivial degree in \( \text{cd}(G/N) \). Again, we have that \( q \) is adjacent to \( p \). It follows that all primes in \( \rho(G) \) are adjacent to \( p \), and hence \( \Delta(G) \) has diameter at most 2.

We now suppose that \( G/N \) is a Frobenius group with \( K/N \) the Frobenius kernel of \( G/N \). We know that \( K/N \) is an elementary abelian \( p \)-group.
for some prime $p$ and $G/K$ is abelian of order $f$ for some integer $f$. If $$r \in \rho(G) - (\pi(f) \cup \{p\}),$$ then $r \in \rho(K)$. It follows that $r$ divides $\theta(1)$ for some $\theta \in \text{Irr}(K)$. By [7, Theorem 12.4], we have that either $f\theta(1) \in \text{cd}(G)$ or $p$ divides $\theta(1)$. It follows that either $r$ is adjacent to all of the primes in $\pi(f)$ or $r$ is adjacent to $p$. Set $$\pi_1 = \{q \in \rho(G) \mid d(q, r) = 1 \text{ for all } r \in \pi(f)\}.$$ If $p \not\in \rho(G)$, then $\rho(G) = \pi(f) \cup \pi_1$, and so the diameter of $\Delta(G)$ is at most 2 and we are finished. We may therefore assume $p \in \rho(G)$ and set $$\pi_2 = \{q \in \rho(G) \mid d(q, p) = 1\}.$$ It follows that $$\rho(G) = \pi(f) \cup \{p\} \cup \pi_1 \cup \pi_2.$$ Observe that each of $\pi(f) \cup \pi_1$ and $\{p\} \cup \pi_2$ induces a subgraph of $\Delta(G)$ of diameter at most 2.

We next claim that $\pi_2 \cap \pi(f)$ and $\pi_1 \cap \pi_2$ cannot both be empty. Suppose to the contrary that $\pi_2 \cap \pi(f) = \emptyset$ and $\pi_1 \cap \pi_2 = \emptyset$. In particular, this implies $p$ is not in $\pi_1$ and $p$ is not adjacent to any prime in $\pi(f) \cup \pi_1$. Since $\Delta(G)$ is connected, there must be a prime $q \in \pi_2$ which is adjacent to some prime $s \in \pi(f) \cup \pi_1$. Hence, there is a character $\chi \in \text{Irr}(G)$ so that $qs$ divides $\chi(1)$. Let $\theta$ be an irreducible constituent of $\chi_K$, and observe that $q$ divides $\theta(1)$. Since $q$ is not in $\pi_1$, we see that $f\theta(1)$ is not in $\text{cd}(G)$, so $p$ divides $\theta(1)$ by [7, Theorem 12.4]. But this is a contradiction since $\theta(1)$ divides $\chi(1)$, and we know that $p$ is not adjacent to $s$. Hence one of $\pi_2 \cap \pi(f)$ or $\pi_1 \cap \pi_2$ is not empty.

Suppose first that $\pi_2 \cap \pi(f)$ is not empty. Let $r' \in \pi_2 \cap \pi(f)$. If $r \in \pi(f) \cup \pi_1$, then $r$ is adjacent to $r'$, and $r'$ is adjacent to $p$. Thus, the distance from $r$ to $p$ is at most 2, and the distance from $r$ to any prime in $\pi_2$ is at most 3. We deduce that $\Delta(G)$ has diameter at most 3 in this case.

Assume now that $\pi_1 \cap \pi_2$ is not empty. This implies that $p$ has a distance at most 3 to all primes in $\pi(f) \cup \pi_1$ and the primes in $\pi(f)$ have a distance at most 3 to all primes in $\pi_2$. The remaining case to consider is $s \in \pi_1$ and $q \in \pi_2$. Notice that $q, s \in \rho(K)$. Since $G$ is not solvable and $G/K$ is abelian, it follows that $K$ is not solvable. By the inductive hypothesis, we have that $\Delta(K)$ has diameter at most 3.

If $\Delta(K)$ is connected, then the distance from $q$ to $s$ in $\Delta(K)$ is at most 3. Since $\Delta(K)$ is a subgraph of $\Delta(G)$, it follows that the distance from $q$ to $s$ in $\Delta(G)$ is at most 3.

The other possibility is that $\Delta(K)$ is disconnected. Using [10], we see that there are characteristic subgroups $L$ and $M$ in $K$ so that $M/L \cong \mathbb{Z}$.
PSL\(2(u)\) for some prime power \(u \geq 4\). To see this, take \(M = K'\) and let \(L\) be the product of all the solvable normal subgroups of \(M\).

Let \(C/L = C_{G/L}(M/L)\). We know that \(G/C M\) is isomorphic to a subgroup of the outer automorphism group of \(M/L\), which is abelian. We have
\[
M = K' \subseteq G' \subseteq C M.
\]
This implies \(G' = M(C \cap G')\). We also know that \(M = K' \subseteq N\), and
\[
K = G' N = (C \cap G') M N = (C \cap G') N.
\]
Let \(A = C \cap G'\), \(B = A \cap N\), and \(D = N \cap G'\). We have \(K/B = N/B \times A/B\).

Observe that
\[
B \cap M = (A \cap N) \cap M = A \cap M = (C \cap G') \cap M = C \cap M = L
\]
and
\[
D = N \cap G' = N \cap A M = (N \cap A) M = BM,
\]

hence \(D/B \cong M/L\).

Observe that \(\rho(G) = \pi(f) \cup \rho(K)\). In the second paragraph of the proof of [11, Lemma 3.2], we proved that \(\rho(K) = \rho(K/(K \cap C))\). Since \(B \subseteq K \cap C\), it follows that \(\rho(K/B) = \rho(K)\). Also, \(A/B \cong K/N\), which is abelian, so \(\rho(N/B) = \rho(K/B) = \rho(K)\). If \(\lambda \in \text{Irr}(K/N)\) is nonprincipal, then \(\lambda^G\) is irreducible. Let \(\alpha = \lambda_B\), and observe that \(\lambda = 1_{N/B} \times \alpha\). It follows that \(K\) is the stabilizer in \(G\) of \(\alpha\). Let \(r\) be any prime in \(\rho(K)\).

Since \(\rho(K) = \rho(N/B)\), there must exist a character \(\nu \in \text{Irr}(N/B)\) so that \(r\) divides \(\nu(1)\). Let \(\theta = \nu \times \alpha \in \text{Irr}(K/B)\). It is easy to see that the stabilizer of \(\theta\) in \(G\) will be \(K\). This implies that both \(r\) and \(f = |G : K|\) will divide all the degrees in \(\text{cd}(G \mid \theta)\). We conclude that all the primes in \(\rho(K)\) must lie in \(\pi_1\), and hence \(\rho(G) = \pi(f) \cup \pi_1\). This completes the proof of the Main Theorem.

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