

When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test

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Patterns, Sequences, and the Ohio Standards

A theme appearing throughout the Patterns, Functions, and Algebra Standard of the Ohio Academic Content Standards for Mathematics [1] is the ability to extend number sequences and patterns. This begins with the K–2 Benchmark:

- B. Extend sequences of sounds and shapes or simple number patterns, and create and record similar patterns.

The idea continues to be developed in a benchmark at every grade-level through the 8–10 Benchmark:

- A. Generalize and explain patterns and sequences in order to find the next term and the n th term.

Recognition and generalization of patterns is central to most of mathematics. In fact, this could almost be a definition of mathematics. The development of this skill is therefore an integral part of a student's mathematical education.

The problems used to teach and assess this ability are often of the following form:

Find the next number in the sequence $\{1, 4, 9, 16, \dots\}$ and find a formula for the n th term in the sequence.

As exercises in pattern recognition, these types of problems can be useful and fun. They should be used for assessment, however, only with extreme caution, if at all.

The reason for the need for caution is that these are not well-posed problems in the sense that there is not a single correct answer. In the example above, the “most obvious” answer is probably that the next number is 25 and the n th term in the sequence is n^2 . But the next number in the sequence could just as easily be 27. Starting the sequence with 1 and

adding the n th odd prime to the n th term to obtain the $n+1$ st term results in the sequence $\{1, 4, 9, 16, 27, 40, 57, \dots\}$.

It is plausible that there simply were not enough terms given in order to distinguish this sequence from the sequence of perfect squares, or that there may be only a few possibilities from which to choose. In fact, we will see that regardless of the number of terms provided, the next term is not uniquely determined. Moreover, given *any* integer b , a polynomial formula for the n th term of the sequence $\{1, 4, 9, 16, \dots\}$ can be found such that b is the fifth term of the sequence.

An Ohio Graduation Test Problem

This becomes a serious issue when such problems are used for assessment, particularly on such high-stakes tests as the Ohio Graduation Test (OGT). The first problem on the Spring 2004 Ohio Graduation Test [2] was as follows.

1. The table below contains the results of a biology experiment.

Record of Blooms

Week	1	2	3	4	5
Number of Blooms	3	9	27	81	b

Assuming the pattern shown in the table continues, what is the value of b ?

- (A) 108
- (B) 130
- (C) 162
- (D) 243

According to the answer key, the correct answer is (D) 243. But there are patterns that will make each of the other answers correct as well. That is, for each answer choice, there is a polynomial function f such that the n th term in the sequence is $f(n)$ and $f(5)$ is the given answer.

Specifically, we have the following. If

$$f_1(n) = -\frac{1}{8}(29n^4 - 322n^3 + 1159n^2 - 1706n + 816),$$

then $f_1(1) = 3$, $f_1(2) = 9$, $f_1(3) = 27$, $f_1(4) = 81$, and $f_1(5) = 108$. If

$$f_2(n) = -\frac{1}{24}(65n^4 - 746n^3 + 2707n^2 - 4018n + 1920),$$

then $f_2(1) = 3$, $f_2(2) = 9$, $f_2(3) = 27$, $f_2(4) = 81$, and $f_2(5) = 130$. If

$$f_3(n) = -\frac{1}{8}(11n^4 - 142n^3 + 529n^2 - 806n + 384),$$

then $f_3(1) = 3$, $f_3(2) = 9$, $f_3(3) = 27$, $f_3(4) = 81$, and $f_3(5) = 162$. Finally, although the n th term that was intended by the authors was probably $f(n) = 3^n$, it is also true that if

$$f_4(n) = 2n^4 - 16n^3 + 52n^2 - 68n + 33,$$

then $f_4(1) = 3$, $f_4(2) = 9$, $f_4(3) = 27$, $f_4(4) = 81$, and $f_4(5) = 243$.

These polynomial functions may not be as “simple” as the exponential function $f(n) = 3^n$, but polynomial functions certainly represent “patterns.” Therefore *any* of the four choices given for the answer to this problem would have to be considered correct. In fact, any non-negative integer whatsoever would have to be considered a correct answer to this problem, as we will now verify.

Every Answer is Correct!

We claim that any integer b is the fifth term in the sequence $\{3, 9, 27, 81, \dots\}$ for some pattern. That is, there is a polynomial

$$f(n) = a_0 + a_1n + a_2n^2 + a_3n^3 + a_4n^4,$$

with rational coefficients, such that $f(1) = 3$, $f(2) = 9$, $f(3) = 27$, $f(4) = 81$, and $f(5) = b$. This translates into the system of linear equations

$$\begin{aligned} a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 + a_4 \cdot 1^4 &= 3 \\ a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 + a_4 \cdot 2^4 &= 9 \\ a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + a_4 \cdot 3^4 &= 27 \\ a_0 + a_1 \cdot 4 + a_2 \cdot 4^2 + a_3 \cdot 4^3 + a_4 \cdot 4^4 &= 81 \\ a_0 + a_1 \cdot 5 + a_2 \cdot 5^2 + a_3 \cdot 5^3 + a_4 \cdot 5^4 &= b. \end{aligned}$$

This system of equations is equivalent to the matrix equation $M\mathbf{a} = \mathbf{s}$, where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \text{and} \quad \mathbf{s} = \begin{bmatrix} 3 \\ 9 \\ 27 \\ 81 \\ b \end{bmatrix}.$$

It is routine to check that the matrix M has a non-zero determinant. Therefore the matrix equation (and hence the system of linear equations) has the unique solution $\mathbf{a} = M^{-1}\mathbf{s}$ for any value of b . Because the entries of M and \mathbf{s} are integers, the entries of \mathbf{a} (the coefficients of f) are rational. The coefficients of the polynomials given above for the cases $b = 108$, $b = 130$, $b = 162$, and $b = 243$ were computed easily using a TI-81 graphing calculator.

It is also interesting to note that the situation is not improved by increasing the number of terms of the sequence that are given or the number

of terms to be determined. Suppose the first m terms $r_1, r_2, r_3, \dots, r_m$ of a sequence of integers are given and we want to find the next k terms in the sequence. If $b_1, b_2, b_3, \dots, b_k$ are any integers and $\ell = m + k$, then there is a polynomial

$$f(n) = a_0 + a_1n + a_2n^2 + \dots + a_{\ell-1}n^{\ell-1},$$

with rational coefficients, such that

$$f(n) = r_n \text{ for } n = 1, \dots, m \text{ and } f(m + j) = b_j \text{ for } j = 1, \dots, k.$$

As in the previous example, this is a system of linear equations in the coefficients $a_0, \dots, a_{\ell-1}$, and we can find the polynomial f by solving for the coefficients. The system of equations is equivalent to the matrix equation $M\mathbf{a} = \mathbf{s}$, where

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{\ell-1} \\ 1 & 3 & 3^2 & \dots & 3^{\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \ell & \ell^2 & \dots & \ell^{\ell-1} \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{\ell-1} \end{bmatrix}, \quad \text{and } \mathbf{s} = \begin{bmatrix} r_1 \\ \vdots \\ r_m \\ b_1 \\ \vdots \\ b_k \end{bmatrix}.$$

To see that the matrix M is non-singular, note that if M is singular, then there is a non-zero vector

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\ell-1} \end{bmatrix}$$

with rational entries such that $M\mathbf{c} = \mathbf{0}$. But then

$$g(x) = c_0 + c_1x + c_2x^2 + \dots + c_{\ell-1}x^{\ell-1}$$

is a non-zero polynomial of degree $\ell - 1$ with ℓ distinct roots $1, 2, \dots, \ell$, which is impossible. (The matrix M is an example of a Vandermonde matrix.) Hence the matrix equation has a unique solution $\mathbf{a} = M^{-1}\mathbf{s}$. Since M and \mathbf{s} have integer entries, the entries of \mathbf{a} (the coefficients of f) are rational.

The problem of finding a polynomial that takes on given values at a finite number of specified points is very old. Many methods have been developed for finding such polynomials, some of which are possibly more elementary than the method described here. These include, for example, the Lagrange interpolation formula and the Newton interpolation formula (see [3, §5.3] or [4]). Such formulas are very important for approximating functions with polynomials and for certain numerical integration techniques.

Conclusions

We have seen that given any number of terms of an integer sequence, there is a “pattern” such that any arbitrary finite sequence of integers will be the next terms in the sequence. The problem to find the “next term” is therefore not a well-posed problem and does not have a unique correct answer. In fact, in a sense, every answer is correct.

There are at least two flaws in the statement of the OGT problem that lead to the concerns discussed here. First, since the problem is multiple choice, there is no way to allow for alternative answers that may be mathematically valid given a reasonable explanation. As an alternative, the question could be more open-ended and require an explanation of the answer. Any answer that can be supported mathematically could then be accepted. The second, more substantial, problem is that the term “pattern” is too loosely defined to be of any real use mathematically. The possibility of different interpretations of pattern leads to the ambiguity that allows for multiple correct answers. In order to avoid this ambiguity, the type of pattern desired would have to be specified.

As mentioned previously, this type of problem does have value as a teaching tool. However, problems for which there are no incorrect answers are not appropriate for the Ohio Graduation Test, and the policy of accepting as correct only one answer for such problems is not mathematically sound. Other than stating the specific type of pattern desired, there is probably no way to state a problem like the OGT problem discussed here that would result in a unique correct answer. Certainly this problem is not specific enough as stated to distinguish between the four answer choices given, and any of the answers (A) - (D) would be mathematically correct. It therefore seems prudent to avoid including such problems on any assessment tool, including the Ohio Graduation Test.

REFERENCES

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- [4] Eric W. Weisstein (1999-2005), “Lagrange Interpolating Polynomial,” from *MathWorld – A Wolfram Web Resource*, <http://mathworld.wolfram.com/LagrangeInterpolatingPolynomial.html>.