Nonsolvable Groups with No Prime Dividing Three Character Degrees

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1 Introduction

Throughout this note, G will be a finite group, $\mathrm{Irr}(G)$ will be the set of irreducible characters of G, and $\mathrm{cd}(G)$ will be the set of character degrees of G. We consider groups where no prime divides at least three degrees in $\mathrm{cd}(G)$. Benjamin studied this question for solvable groups in [1]. She proved that solvable groups with this property satisfy $|\mathrm{cd}(G)| \leqslant 6$. She also presented examples to show that this bound is met. McVey has a different family of examples in [14].

We now consider this question for nonsolvable groups. We begin by classifying all simple and almost simple groups with the property that no prime divides three degrees.

Theorem 1. *Let S be a finite simple group and G a group such that $S \leqslant G \leqslant \mathrm{Aut}\, S$. No prime divides three degrees of G if and only if $S \cong \mathrm{PSL}_2(q)$, $q \geqslant 4$ a prime power, and one of the following holds:*

1. $G = S \cong \mathrm{PSL}_2(q)$,
2. $G \cong \mathrm{PGL}_2(q)$, *$q$ odd,*
3. $G \cong \mathrm{PSL}_2(3^f) \rtimes Z_f$, *where $f \neq 3$ is a prime,*
4. $G \cong \mathrm{PSL}_2(2^f) \rtimes Z_f$, *where f is a prime,*
5. $G \cong \mathrm{PSL}_2(2^f) \rtimes Z_r$, *where $r < f$ is an odd prime divisor of f with $r \nmid 2^f - 1$ and $r \nmid 2^f + 1$.*

The character degree sets in these cases are as in Table 1.

We then consider general nonsolvable groups. We will prove that if G is a nonsolvable group where no prime divides three degrees in $\mathrm{cd}(G)$, then there is a solvable normal subgroup L so that G/L is an almost simple group where no prime divides three degrees in $\mathrm{cd}(G/L)$. Hence, we can study these groups based on Theorem 1. In particular, G can be associated with a unique simple group of the form $S \cong \mathrm{PSL}_2(q)$, for a prime power $q \geqslant 4$, and $S \leqslant G/L \leqslant \mathrm{Aut}\, S$. If $q > 5$ is odd or G/L properly contains $\mathrm{PSL}_2(2^f)$, then we prove that $\mathrm{cd}(G) = \mathrm{cd}(G/L)$ and $L = Z(G)$. If $G/L \cong \mathrm{PSL}_2(2^f)$, then $|\mathrm{cd}(G)| \leqslant |\mathrm{cd}(G/L)| + 1 = 5$. In all nonsolvable cases, we have $|\mathrm{cd}(G)| \leqslant 6$, and there are examples where the bound is met (see Table 1). Combined with Benjamin's result, we obtain the following theorem.

Theorem 2. *If G is any group where no prime divides three degrees, then $|\mathrm{cd}(G)| \leqslant 6$.*

Table 1: Degrees of Almost Simple Groups with No Prime Dividing Three Degrees

Group G	$\mathrm{cd}(G)$
$\mathrm{PSL}_2(q)$, $q > 5$ odd	$\{1, q-1, q, q+1, (q+\epsilon)/2\}$
$\mathrm{PSL}_2(q)$, $q \geqslant 4$ even	$\{1, q-1, q, q+1\}$
$\mathrm{PGL}_2(q)$, q odd	$\{1, q-1, q, q+1\}$
$\mathrm{PSL}_2(3^2) \rtimes Z_2 \cong \mathrm{PGL}_2(9)$	$\{1, 8, 9, 10\}$
$\mathrm{PSL}_2(3^2) \rtimes Z_2 \cong M_{10}$	$\{1, 9, 10, 16\}$
$\mathrm{PSL}_2(3^2) \rtimes Z_2 \cong S_6$	$\{1, 5, 9, 10, 16\}$
$\mathrm{PSL}_2(3^f) \rtimes Z_f$, $f > 3$ prime	$\{1, 3^f, (3^f-1)f, (3^f+1)f, (3^f-1)/2\}$
$\mathrm{PSL}_2(2^2) \rtimes Z_2 \cong S_5$	$\{1, 4, 5, 6\}$
$\mathrm{PSL}_2(2^f) \rtimes Z_f$, $f > 2$ prime	$\{1, 2^f-1, 2^f, (2^f-1)f, (2^f+1)f\}$
$\mathrm{PSL}_2(2^f) \rtimes Z_r$, r odd prime, $r \mid f$, $r < f$	$\{1, 2^f-1, 2^f, 2^f+1, (2^f-1)r, (2^f+1)r\}$

$\epsilon = (-1)^{(q-1)/2}$

This is related to a question about character degree graphs. We define the degree-vertex graph of G to be $\Gamma(G)$, whose vertex set is $\mathrm{cd}(G) \setminus \{1\}$ and there is an edge between a and b if $(a, b) > 1$. Facts about this graph can be found in [9].

We wish to study the groups G where $\Gamma(G)$ contains no triangles. Observe that if $\Gamma(G)$ contains no triangles, then no prime divides three degrees in $\mathrm{cd}(G)$. We then obtain the following.

Theorem 3. *If G is any group and $\Gamma(G)$ contains no triangles, then $|\mathrm{cd}(G)| \leqslant 6$.*

We note that the examples of Benjamin in [1] and McVey in [14] have degree-vertex graphs with no triangles, so this bound cannot be improved in the solvable case.

Observe that if G is an almost simple group for which no prime divides three degrees, as given in Table 1, then $\Gamma(G)$ contains no triangles. More generally, we have the following theorem, which shows that the answer is the same as when no prime divides three degrees.

Theorem 4. *Let G be a nonsolvable group. No prime divides three degrees in $\mathrm{cd}(G)$ if and only if $\Gamma(G)$ has no triangles.*

It now seems natural to ask whether there exist groups so that no prime divides three degrees, but $\Gamma(G)$ contains a triangle. We have seen that there are no nonsolvable groups with this property. There do exist solvable groups with the property, however. The first author, Alex Moretó, and Tom Wolf have constructed such solvable groups in Section 6 of [10]. These groups are parametrized by primes p and q so that p is congruent to 1 mod 3 and is not a Mersenne prime, and q is an odd prime divisor of $p+1$. The character degree set is $\{1, 3q, p^2q, 3p^3\}$, hence no prime divides three degrees, but the degree-vertex graph is a triangle.

A related question has been studied by Wu and Zhang in [19] and by Li, Liu, and Song in [11], where solvable and nonsolvable groups, respectively, whose character graphs contain no triangles are classified. The character graph of a finite group G has the set of nonlinear characters in $\mathrm{Irr}(G)$ as its vertex set (as opposed to $\mathrm{cd}(G) \setminus \{1\}$ for the degree-vertex graph $\Gamma(G)$), with an edge between χ and ψ in $\mathrm{Irr}(G)$ if $(\chi(1), \psi(1)) > 1$. The condition on G that this graph is triangle-free is, of course, much stronger than the condition that $\Gamma(G)$ is triangle-free, or even that no prime divides three distinct degrees. Our study of the graph $\Gamma(G)$ is in the context of divisibility properties between distinct character degrees.

2 Simple Groups

In this section we determine the finite simple groups for which no prime divides three distinct degrees. Lemmas 2.1, 2.2, and 2.3 imply that no prime divides three degrees of the simple group G if and only if $G \cong \mathrm{PSL}_2(q)$ for some prime power q.

2.1 Sporadic and Alternating Groups

Lemma 2.1. *If G is a sporadic simple group, then there is a prime that divides three degrees of G.*

Proof. Table 2 lists three distinct even degrees for each sporadic group G. Notation for the characters is as in the Atlas [4]. □

Lemma 2.2. *Let G be the simple alternating group A_n for $n \geqslant 5$. No prime divides three degrees of G if and only if $n = 5$ or $n = 6$.*

Proof. We have $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$ and $\mathrm{cd}(A_6) = \{1, 5, 8, 9, 10\}$, hence no prime divides three degrees of G for $n = 5, 6$. The group A_7 has even character degrees 6, 10, and 14.

We now assume $G = A_n$ with $n \geqslant 8$, and consider the irreducible character $\chi_{r,s}$ of the symmetric group S_n corresponding to the partition $(n - s - r, s + 1, 1^{r-1})$. As shown in [12], this partition exists provided $r \geqslant 1$, $s \geqslant 0$, and $r + 2s + 1 \leqslant n$, in which case

$$\chi_{r,s}(1) = \binom{n}{s}\binom{n-s-1}{r-1}\frac{n-2s-r}{r+s}.$$

Moreover, $\chi_{r,s}$ restricts irreducibly to A_n unless either $s = 0$ and $n = 2r+1$ or $s = 1$ and $n = 2r+2$. Hence we have the degrees listed in Table 3 for A_n with $n \geqslant 8$.

Observe that $\chi_{1,1}$, $\chi_{2,1}$, and $\chi_{1,2}$ are even when $n \equiv 0 \pmod 4$, $\chi_{2,0}$, $\chi_{3,0}$, and $\chi_{1,2}$ are even when $n \equiv 1 \pmod 4$, $\chi_{2,0}$, $\chi_{3,0}$, and $\chi_{2,1}$ are even when $n \equiv 2 \pmod 4$, and $\chi_{3,0}$, $\chi_{1,1}$, and $\chi_{1,2}$ are even when $n \equiv 3 \pmod 4$. It is easily verified that since $n \geqslant 8$, these degrees of A_n are all distinct, and so 2 divides three degrees of A_n for all $n \geqslant 7$. □

2.2 Groups of Lie Type

Finally, we consider the groups of Lie type. These are the classical groups:

$$A_1(q) \cong \mathrm{PSL}_2(q), \text{ for } q \neq 2, 3,$$
$$A_\ell(q) \cong \mathrm{PSL}_{\ell+1}(q),\ {}^2A_\ell(q^2) \cong \mathrm{PSU}_{\ell+1}(q^2), \text{ for } \ell \geqslant 2 \text{ (and } q \neq 2 \text{ if } \ell = 2),$$
$$B_\ell(q) \cong \Omega_{2\ell+1}(q),\ C_\ell(q) \cong \mathrm{PSp}_{2\ell}(q), \text{ for } \ell \geqslant 2 \text{ (and } q \neq 2 \text{ if } \ell = 2),$$
$$D_\ell(q) \cong \mathrm{P}\Omega^+_{2\ell}(q),\ {}^2D_\ell(q^2) \cong \mathrm{P}\Omega^-_{2\ell}(q) \text{ for } \ell \geqslant 4,$$

and the groups of exceptional Lie type:

$$G_2(q) \text{ for } q \neq 2,$$
$${}^2G_2(q^2) \text{ for } q^2 \neq 3,\ {}^2B_2(q^2) \text{ for } q^2 \neq 2,$$
$${}^2F_4(q^2) \text{ for } q^2 \neq 2,\ {}^2F_4(2)',$$
$$F_4(q),\ E_6(q),\ E_7(q),\ E_8(q),\ {}^2E_6(q^2),\ {}^3D_4(q^3).$$

The restrictions on ℓ and q are so that the groups will be simple and (generally) not isomorphic to others in the list. These restrictions will always be assumed in what follows.

Table 2: Degrees of Sporadic Groups

Grp.	Chars.	Degrees	Grp.	Chars.	Degrees
M_{11}	χ_3	10	$O'N$	χ_2	10944
	χ_6	16		χ_3	13376
	χ_8	44		χ_5	25916
M_{12}	χ_4	16	Co_3	χ_6	896
	χ_7	54		χ_9	2024
	χ_{11}	66		χ_{10}	3520
J_1	χ_2	56	Co_2	χ_6	2024
	χ_4	76		χ_9	7084
	χ_9	120		χ_{14}	12650
M_{22}	χ_7	154	Fi_{22}	χ_2	78
	χ_8	210		χ_5	1430
	χ_{10}	280		χ_7	3080
J_2	χ_2	14	HN	χ_4	760
	χ_6	36		χ_5	3344
	χ_8	70		χ_6	8778
M_{23}	χ_2	22	Ly	χ_2	2480
	χ_5	230		χ_4	45694
	χ_{10}	770		χ_5	48174
HS	χ_2	22	Th	χ_2	248
	χ_4	154		χ_4	27000
	χ_{10}	770		χ_6	30628
J_3	χ_6	324	Fi_{23}	χ_2	782
	χ_7	646		χ_3	3588
	χ_9	816		χ_5	25806
M_{24}	χ_7	252	Co_1	χ_2	276
	χ_{10}	770		χ_6	17250
	χ_{12}	990		χ_7	27300
M^cL	χ_2	22	J_4	χ_6	887778
	χ_4	252		χ_{11}	1776888
	χ_5	770		χ_{21}	95288172
He	χ_6	680	Fi'_{24}	χ_4	249458
	χ_{12}	1920		χ_{13}	48893768
	χ_{13}	4080		χ_{14}	74837400
Ru	χ_2	378	B	χ_4	1139374
	χ_4	406		χ_5	9458750
	χ_6	3276		χ_8	347643114
Suz	χ_4	780	M	χ_3	21296876
	χ_6	3432		χ_4	842609326
	χ_9	5940		χ_5	18538750076

Table 3: Degrees of A_n, $n \geqslant 8$

Char.	Degree
$\chi_{2,0}$	$\frac{(n-1)(n-2)}{2}$
$\chi_{3,0}$	$\frac{(n-1)(n-2)(n-3)}{2\cdot 3}$
$\chi_{1,1}$	$\frac{n(n-3)}{2}$
$\chi_{2,1}$	$\frac{n(n-2)(n-4)}{3}$
$\chi_{1,2}$	$\frac{n(n-1)(n-5)}{2\cdot 3}$

Lemma 2.3. *Let G be a simple group of Lie type. No prime divides three degrees of G if and only if $G \cong \mathrm{PSL}_2(q)$ for some prime power q.*

Proof. Suppose first that $G \cong \mathrm{PSL}_2(q)$. If $q = 2^f$ for $f \geqslant 2$, then

$$\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, 2^f + 1\}$$

and distinct degrees are coprime. Observe that this includes the case $G = \mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4) \cong A_5$. If $q > 5$ is odd, then

$$\mathrm{cd}(G) = \{1, q-1, q, q+1, (q+\epsilon)/2\},$$

where $\epsilon = (-1)^{(q-1)/2}$. The degrees 1 and q are coprime to all other degrees. Now $(q-1, q+1) = 2$, but $(q+\epsilon)/2$ is odd, and so the degrees $q-1$, $q+1$, and $(q+\epsilon)/2$ cannot have a common prime factor.

Assume now that $G = G(q)$ is a simple group of Lie type over a field of characteristic p, so q is a power of p, but G is not isomorphic to $\mathrm{PSL}_2(q)$. If $G \cong \mathrm{PSL}_3(4)$, then G has character degrees 20, 35, and 45 (see the Atlas [4]), which are all divisible by 5. Similarly, if $G \cong {}^2F_4(2)'$, then G has even degrees 26, 78, and 300.

If $G \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$, $m \geqslant 1$, then Table 6 lists three distinct degrees of G divisible by $\Phi_1\Phi_2 = q^2 - 1$ (see [18]). As $q^2 \neq 2$, these degrees are all divisible by a common prime.

For all other cases, three distinct character degrees divisible by p are given in Table 4 (classical groups of rank greater than 2), Table 5 (groups of exceptional type of rank greater than 2), or Table 6 (groups of rank 2). The degrees in Tables 4 and 5 are of unipotent characters and can be found in [3]. The degrees in Table 6 are from [3], [16], and [18]. □

3 Almost Simple Groups

In this section, we assume S is a finite simple group and G is a group satisfying $S \leqslant G \leqslant \mathrm{Aut}\, S$. We show via Lemmas 3.1, 3.3, and 3.4 that if G is not isomorphic to $\mathrm{PSL}_2(q)$, then there is a prime that divides three degrees of G.

If $\chi \in \mathrm{Irr}(S)$ and $\hat{\chi} \in \mathrm{Irr}(G)$ lies over χ, then $\hat{\chi}(1) = a\chi(1)$ for some integer a dividing $|G : S|$, by Corollary 11.29 of [8]. In particular, $\chi(1) \mid \hat{\chi}(1)$, and so if $\chi_1(1)$, $\chi_2(1)$, $\chi_3(1)$ have a common prime divisor, then $\hat{\chi}_1(1)$, $\hat{\chi}_2(1)$, $\hat{\chi}_3(1)$ are also divisible by that prime. Thus a prime divides three degrees of G unless two of these degrees of G are equal; that is, unless $a\chi_i(1) = b\chi_j(1)$ for some divisors a and b of $|G : S|$.

Table 4: Degrees of Groups of Classical Lie Type

Group	Labels	Degrees
$A_\ell(q)$, $\ell \geqslant 3$	St	$q^{\ell(\ell+1)/2}$
	$(1, \ell)$	$q \cdot \dfrac{q^\ell - 1}{q - 1}$
	$(1, 1, \ell - 1)$	$q^3 \cdot \dfrac{(q^{\ell-1} - 1)(q^\ell - 1)}{(q - 1)(q^2 - 1)}$
${}^2A_\ell(q^2)$, $\ell \geqslant 3$	St	$q^{\ell(\ell+1)/2}$
	$(1, \ell)$	$q \cdot \dfrac{q^\ell - (-1)^\ell}{q + 1}$
	$(1, 1, \ell - 1)$	$q^3 \cdot \dfrac{(q^{\ell-1} - (-1)^{\ell-1})(q^\ell - (-1)^\ell)}{(q + 1)(q^2 - 1)}$
$B_\ell(q), C_\ell(q)$, $\ell \geqslant 3$	St	q^{ℓ^2}
	$\begin{pmatrix} 0 & \ell - 1 \\ & 2 \end{pmatrix}$	$\dfrac{1}{2} q^2 \cdot \dfrac{(q^{\ell-3} + 1)(q^{\ell-1} - 1)(q^{2\ell} - 1)}{(q^2 - 1)^2}$
	$\begin{pmatrix} 1 & & 2 & & \ell \\ & 0 & & 1 & \end{pmatrix}$	$\dfrac{1}{2} q^4 \cdot \dfrac{(q^{\ell-2} - 1)(q^{2(\ell-1)} - 1)(q^\ell + 1)}{(q^2 - 1)^2}$
$D_\ell(q)$, $\ell \geqslant 4$	St	$q^{\ell(\ell-1)}$
	$\begin{pmatrix} \ell - 1 \\ 1 \end{pmatrix}$	$q \cdot \dfrac{(q^{\ell-2} + 1)(q^\ell - 1)}{q^2 - 1}$
	$\begin{pmatrix} 1 & 2 & \ell \\ 0 & 1 & 2 \end{pmatrix}$	$q^6 \cdot \dfrac{(q^{2(\ell-2)} - 1)(q^{2(\ell-1)} - 1)}{(q^2 - 1)(q^4 - 1)}$
${}^2D_\ell(q^2)$, $\ell \geqslant 4$	St	$q^{\ell(\ell-1)}$
	$\begin{pmatrix} 1 & \ell - 1 \\ & - \end{pmatrix}$	$q \cdot \dfrac{(q^{\ell-2} - 1)(q^\ell + 1)}{q^2 - 1}$
	$\begin{pmatrix} 0 & 1 & 2 & \ell \\ & 1 & 2 & \end{pmatrix}$	$q^6 \cdot \dfrac{(q^{2(\ell-2)} - 1)(q^{2(\ell-1)} - 1)}{(q^2 - 1)(q^4 - 1)}$

Table 5: Degrees of Groups of Exceptional Lie Type

Group	Labels	Degrees
${}^3D_4(q^3)$	$\phi_{1,6}$	q^{12}
	$\phi'_{1,3}$	$q\Phi_{12}$
	$\phi''_{1,3}$	$q^7\Phi_{12}$
${}^2F_4(q^2)$	ε	q^{24}
	ε'	$q^2\Phi_{12}\Phi_{24}$
	ε''	$q^{10}\Phi_{12}\Phi_{24}$
$F_4(q)$	$\phi_{1,24}$	q^{24}
	$\phi_{9,2}$	$q^2\Phi_3^2\Phi_6^2\Phi_{12}$
	$\phi_{9,10}$	$q^{10}\Phi_3^2\Phi_6^2\Phi_{12}$
$E_6(q)$	$\phi_{1,36}$	q^{36}
	$\phi_{6,1}$	$q\Phi_8\Phi_9$
	$\phi_{6,25}$	$q^{25}\Phi_8\Phi_9$
${}^2E_6(q^2)$	$\phi_{1,24}$	q^{36}
	$\phi'_{2,4}$	$q\Phi_8\Phi_{18}$
	$\phi''_{2,16}$	$q^{25}\Phi_8\Phi_{18}$
$E_7(q)$	$\phi_{1,63}$	q^{63}
	$\phi_{7,1}$	$q\Phi_7\Phi_{12}\Phi_{14}$
	$\phi_{7,46}$	$q^{46}\Phi_7\Phi_{12}\Phi_{14}$
$E_8(q)$	$\phi_{1,120}$	q^{120}
	$\phi_{8,1}$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
	$\phi_{8,91}$	$q^{91}\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$

Table 6: Degrees of Groups of Rank 2

Group	Labels	Degrees
$A_2(q),\ q \neq 2, 4$	St	q^3
	χ_{qs}	$q(q+1)$
	χ_{qt}	$q(q^2+q+1)$
${}^2A_2(q^2),\ q \neq 2$	St	q^3
	χ_{qs}	$q(q-1)$
	χ_{qt}	$q(q^2-q+1)$
$B_2(q), C_2(q),\ q \neq 2$	St	q^4
	$\binom{0\ \ 2}{1}$	$\frac{1}{2}q(q+1)^2$
	$\binom{1\ \ 2}{0}$	$\frac{1}{2}q(q^2+1)$
${}^2B_2(q^2),\ q^2 \neq 2$		$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2$
		$\Phi_1\Phi_2\Phi_8'$
		$\Phi_1\Phi_2\Phi_8''$
$G_2(q),\ q \neq 2$	$\phi_{1,6}$	q^6
	$G_2[-1]$	$\frac{1}{2}q\Phi_1^2\Phi_3$
	$\phi_{2,2}$	$\frac{1}{2}q\Phi_2^2\Phi_6$
${}^2G_2(q^2),\ q^2 \neq 3$	ε	q^6
	cusp	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi_{12}'$
	cusp	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi_{12}''$

Lemma 3.1. *If S is a sporadic simple group or an alternating group A_n with $n \geqslant 7$, and $S \leqslant G \leqslant \operatorname{Aut} S$, then there is a prime that divides three degrees of G.*

Proof. First assume S is a sporadic group, hence $|\operatorname{Aut} S : S| \leqslant 2$ and Table 2 lists three even character degrees of S. Thus if $G \neq S$, then $|G : S| = 2$. Observe that in all cases, no character degree given is 2 times another of the degrees. Hence, by the remarks above, degrees of G that lie over those of S shown in Table 2 are even and distinct.

If $S = A_n$ with $n \geqslant 7$, then $|\operatorname{Aut} S : S| = 2$. We may therefore assume $G = \operatorname{Aut} S \cong S_n$. The Atlas [4] shows that $G \cong S_7$ has even degrees 6, 14, and 20. For $n \geqslant 8$, the degrees of A_n shown in Table 3 are in fact restrictions of irreducible characters of $G \cong S_n$, and so S_n also has three even degrees. □

We should note that $A_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ and $A_6 \cong \mathrm{PSL}_2(9)$. These groups will be considered along with the groups $\mathrm{PSL}_2(q)$ later.

We next consider the case where $S = S(q)$ is a simple group of Lie type of rank greater than 2 over a field of q elements of characteristic p. In this case, Tables 4 and 5 list three degrees of S that are divisible by p. In order to prove that degrees of G lying over these degrees of S are distinct, we will require the following elementary lemma.

Lemma 3.2. *If $q = p^f$, where p is prime and f is a positive integer, then $q > 2f$ unless $p = 2$ and $f \leqslant 2$, in which case $q = 2f$.*

Proof. If $q = 2$ or $q = 2^2$, then clearly $q = 2f$. Thus we assume that if $p = 2$, then $f \geqslant 3$. Observe that in general $p^f - 1 = (p-1)(p^{f-1} + p^{f-2} + \cdots + p^2 + p + 1)$.

If $p \geqslant 3$, then

$$\begin{aligned} p^f &= (p-1)(p^{f-1} + p^{f-2} + \cdots + p^2 + p + 1) + 1 \\ &\geqslant 2(1 + 1 + \cdots + 1 + 1 + 1) + 1 \\ &= 2f + 1. \end{aligned}$$

Hence $q = p^f > 2f$ in this case.

If $p = 2$, then we have $f \geqslant 3$. We have

$$\begin{aligned} 2^f &= (2-1)(2^{f-1} + 2^{f-2} + \cdots + 2^2 + 2 + 1) + 1 \\ &= 2^{f-1} + 2^{f-2} + \cdots + 2^2 + 2 + 2 \\ &= 2(2^{f-2} + 2^{f-3} + \cdots + 2 + 1 + 1) \\ &\geqslant 2(1 + 1 + \cdots + 2 + 1 + 1) \\ &= 2(f+1). \end{aligned}$$

Hence $q = 2^f > 2f$ again in this case. □

Lemma 3.3. *If S is a simple group of Lie type of rank greater than 2 and $S \leqslant G \leqslant \operatorname{Aut} S$, then there is a prime that divides three degrees of G.*

Proof. If $S \cong {}^2F_4(2)'$, then $|\operatorname{Aut} S : S| = 2$, and so $G = S$ or $G = \operatorname{Aut} S$. By the Atlas [4] character table, $\operatorname{Aut} S$ has even degrees 52, 78, and 300.

Now let $S = S(q)$ be a simple group of Lie type of rank greater than 2 over a field of q elements of characteristic p (other than ${}^2F_4(2)'$). Three degrees of S are given in Table 4 or Table 5. In each case, one of these degrees is that of the Steinberg character St. Denote the other two characters by χ_1 and χ_2, in the order listed in the tables. The degree of each of these characters is divisible by p.

Let $\hat{\mathrm{St}}$, $\hat{\chi}_1$, and $\hat{\chi}_2$ be irreducible characters of G lying over St, χ_1, and χ_2, respectively. The degree of each of these is divisible by p, and so there is a prime dividing three degrees of G unless two of the degrees are equal. The Steinberg character St extends to an irreducible character of $\operatorname{Aut} S$ (see [15]), hence $\hat{\mathrm{St}}(1)$ is a power of q. Both $\chi_1(1)$ and $\chi_2(1)$ are divisible by primes other than p, and so $\hat{\mathrm{St}}(1)$ cannot be equal to either of $\hat{\chi}_1(1)$ or $\hat{\chi}_2(1)$.

Observe that $\chi_2(1)$ has a greater p-part than $\chi_1(1)$ in each case, and so we have $(\chi_2(1))_p = p^\alpha(\chi_1(1))_p$ for some positive integer α. As noted previously, $\hat{\chi}_1(1) = a\chi_1(1)$ and $\hat{\chi}_2(1) = b\chi_2(1)$ for some integers a, b dividing $|G : S|$, hence also $|\operatorname{Aut} S : S|$. Therefore, if $\hat{\chi}_1(1) = \hat{\chi}_2(1)$, then p^α must be a divisor of $|\operatorname{Aut} S : S|$.

In Atlas [4] notation, $|\operatorname{Aut} S : S| = dfg$, where d, f, and g are the orders of the groups of diagonal, field, and graph automorphisms, respectively, modulo inner automorphisms. In particular, $(d, q) = 1$ in all cases and $g \leqslant 2$ unless $S \cong D_4(q)$, in which case $g = 6$. The parameter f is defined by $q = p^f$ except when $S \cong {}^3D_4(q^3)$, where $q^3 = p^f$, or when S is one of ${}^2A_\ell(q^2)$, ${}^2D_\ell(q^2)$, ${}^2E_6(q^2)$, or ${}^2F_4(q^2)$, where $q^2 = p^f$.

It follows that $|\operatorname{Aut} S : S|_p$ divides $2f$, except possibly in the case $S \cong D_4(q)$, where $|\operatorname{Aut} S : S|_p$ divides $6f$. Recall that $(\chi_2(1))_p = p^\alpha(\chi_1(1))_p$ and observe that $p^\alpha \geqslant (p^f)^2$, except in the case $S \cong {}^2A_\ell(q^2)$, where $p^\alpha = p^f$. If S is neither $D_4(q)$ nor ${}^2A_\ell(q^2)$, then by Lemma 3.2,

$$p^\alpha \geqslant (p^f)^2 \geqslant p^f \cdot 2f > 2f \geqslant |\operatorname{Aut} S : S|_p.$$

If $S \cong {}^2A_\ell(q^2)$, then $g = 1$ and so $|\operatorname{Aut} S : S|_p$ divides f, hence

$$p^\alpha = p^f \geqslant 2f > f \geqslant |\operatorname{Aut} S : S|_p.$$

If $S \cong D_4(q)$, then $g = 6$, but $(\hat{\chi}_2(1))_p = (p^f)^5(\hat{\chi}_1(1))_p$, and so

$$p^\alpha \geqslant (p^f)^5 \geqslant (p^f)^4 \cdot 2f > 6f \geqslant |\operatorname{Aut} S : S|_p.$$

Hence in all cases, if $(\chi_2(1))_p = p^\alpha(\chi_1(1))_p$, then $p^\alpha \nmid |\operatorname{Aut} S : S|$. Therefore, $\hat{\chi}_1(1) \neq \hat{\chi}_2(1)$ and so $\hat{\mathrm{St}}(1)$, $\hat{\chi}_1(1)$, and $\hat{\chi}_2(1)$ are distinct degrees of G divisible by p. □

Lemma 3.4. *If S is a simple group of Lie type of rank 2 and $S \leqslant G \leqslant \operatorname{Aut} S$, then there is a prime that divides three degrees of G.*

Proof. Let $S = S(q)$ be a simple group of Lie type of rank 2 over a field of q elements of characteristic p. Three degrees of S, for S other than $A_2(4) \cong \mathrm{PSL}_3(4)$, are given in Table 6. Denote the three characters by χ_1, χ_2, and χ_3, in the order listed in the table. Let $\hat{\chi}_1$, $\hat{\chi}_2$, and $\hat{\chi}_3$ be irreducible characters of G lying over χ_1, χ_2, and χ_3, respectively.

The degrees $\hat{\chi}_1(1)$, $\hat{\chi}_2(1)$, $\hat{\chi}_3(1)$ have a common prime factor, hence three degrees of G are divisible by that prime unless two of the degrees are equal. Moreover, if $\hat{\chi}_i(1) = \hat{\chi}_j(1)$, then $a\chi_i(1) = b\chi_j(1)$ for some divisors a, b of $|G : S|$, hence of $|\operatorname{Aut} S : S|$.

If $S \cong \mathrm{PSL}_3(4)$, then S has degrees $\chi_1(1) = 20$, $\chi_2(1) = 35$, and $\chi_3(1) = 45$. In this case, $|\operatorname{Aut} S : S| = 12$. If $a\chi_2(1) = b\chi_j(1)$ for $j = 1, 3$, then $7 \nmid \chi_j(1)$ implies $7 \mid b$, which contradicts $b \mid 12$. If $a\chi_3(1) = b\chi_j(1)$ for $j = 1, 2$, then $(9, \chi_j(1)) = 1$ implies $9 \mid b$, again contradicting $b \mid 12$. Hence $\hat{\chi}_1(1)$, $\hat{\chi}_2(1)$, $\hat{\chi}_3(1)$ are distinct degrees of G divisible by 5. We may now assume S is a simple group of Lie type of rank 2 other than $\mathrm{PSL}_3(4)$.

If $S \cong {}^2B_2(q^2)$, then $\chi_1(1)$ is even, while $\chi_2(1)$ and $\chi_3(1)$ are odd. In this case, $|\operatorname{Aut} S : S| = f$, where $q^2 = 2^{2m+1} = 2^f$. Hence $|\operatorname{Aut} S : S|$ is odd, thus $\hat{\chi}_2(1)$ and $\hat{\chi}_3(1)$ are also odd, and so neither can equal $\hat{\chi}_1(1)$, which is even.

In all other cases, χ_1 is the Steinberg character St, which extends irreducibly to G (see [15]). Hence $\hat{\chi}_1(1)$ is a power of p, whereas $\hat{\chi}_2(1)$ and $\hat{\chi}_3(1)$ are divisible by other primes. Again, this implies that neither $\hat{\chi}_2(1)$ nor $\hat{\chi}_3(1)$ can equal $\hat{\chi}_1(1)$.

We now have that $\hat{\chi}_1(1)$, $\hat{\chi}_2(1)$, $\hat{\chi}_3(1)$ are three degrees of G with a common prime factor unless $\hat{\chi}_2(1) = \hat{\chi}_3(1)$. As noted previously, this would imply $a\chi_2(1) = b\chi_3(1)$ for some divisors a and b of $|\operatorname{Aut} S : S|$. We assume this holds and obtain a contradiction in each case.

Type A_2. If $S \cong A_2(q) \cong \mathrm{PSL}_3(q)$, then $\chi_2(1) = q(q+1)$ and $\chi_3(1) = q(q^2+q+1)$. In this case, $|\operatorname{Aut} S : S| = 2df$, where $d = (3, q-1)$ and $q = p^f$. Since $(q+1, q^2+q+1) = 1$, $a \cdot q(q+1) = b \cdot q(q^2+q+1)$ implies $q+1 \mid b$ and $q^2+q+1 \mid a$. Thus $(q+1)(q^2+q+1)$ divides $|\operatorname{Aut} S : S|$, hence divides $6f$. However, $q > 4$ and $q = p^f \geqslant 2f$ by Lemma 3.2, hence

$$(q+1)(q^2+q+1) > q^3 \geqslant q^2 \cdot 2f > 6f,$$

a contradiction.

Type $^2A_2$. If $S \cong {}^2A_2(q^2) \cong \mathrm{PSU}_3(q^2)$, then $\chi_2(1) = q(q-1)$ and $\chi_3(1) = q(q^2-q+1)$. In this case, $|\operatorname{Aut} S : S| = df$, where $d = (3, q+1)$ and $q^2 = p^f$. Since $(q-1, q^2-q+1) = 1$, $a \cdot q(q-1) = b \cdot q(q^2-q+1)$ implies $q-1 \mid b$ and $q^2-q+1 \mid a$. Thus $(q-1)(q^2-q+1)$ divides $|\operatorname{Aut} S : S|$, hence divides $3f$. However, $q \geqslant 3$ and $q = p^{f/2} \geqslant f$ by Lemma 3.2, hence

$$(q-1)(q^2-q+1) = (q-1)(q(q-1)+1) > (q-1)^2 q > 3f,$$

a contradiction.

Types B_2, C_2. If $S \cong B_2(q) \cong C_2(q) \cong \mathrm{PSp}_4(q)$, then $\chi_2(1) = \frac{1}{2}q(q+1)^2$ and $\chi_3(1) = \frac{1}{2}q(q^2+1)$. In this case, $|\operatorname{Aut} S : S| = 2f$, where $q = p^f$. (If q is odd, then $d = 2$ and $g = 1$, and if q is even, then $d = 1$ and $g = 2$.) We assume $a\chi_2(1) = b\chi_3(1)$, and hence $a(q+1)^2 = b(q^2+1)$.

If q is even, then $((q+1)^2, q^2+1) = 1$, and so $(q+1)^2 \mid b$ and $q^2+1 \mid a$. Thus $(q+1)^2(q^2+1)$ divides $|\operatorname{Aut} S : S| = 2f$. However, $q > 2$ and $q = p^f \geqslant 2f$ by Lemma 3.2, hence

$$(q+1)^2(q^2+1) > q^4 \geqslant q^3 \cdot 2f > 2f,$$

a contradiction.

If q is odd, then $((q+1)^2, q^2+1) = 2$, and so $\frac{1}{2}(q+1)^2 \mid b$ and $\frac{1}{2}(q^2+1) \mid a$. Thus $\frac{1}{4}(q+1)^2(q^2+1)$ divides $|\operatorname{Aut} S : S| = 2f$. However, $q > 2$ and $q = p^f \geqslant 2f$ by Lemma 3.2, hence

$$\frac{1}{4}(q+1)^2(q^2+1) > \frac{1}{4}q^4 \geqslant \frac{1}{4}q^3 \cdot 2f > 2f,$$

a contradiction.

Type $^2B_2$. Let $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ with $m \geqslant 1$. Let $\Phi_8' = q^2 + \sqrt{2}q + 1$ and $\Phi_8'' = q^2 - \sqrt{2}q + 1$, so that $\Phi_8'\Phi_8'' = \Phi_8 = q^4+1$. We have $\chi_2(1) = \Phi_1\Phi_2\Phi_8'$ and $\chi_3(1) = \Phi_1\Phi_2\Phi_8''$. In this case, $|\operatorname{Aut} S : S| = f$, where $q^2 = 2^f$; that is, $f = 2m+1$. Since $(\Phi_8', \Phi_8'') = 1$,

$$a \cdot \Phi_1\Phi_2\Phi_8' = b \cdot \Phi_1\Phi_2\Phi_8''$$

implies $\Phi_8' \mid b$ and $\Phi_8'' \mid a$. Thus $\Phi_8'\Phi_8'' = \Phi_8 = q^4+1$ divides $|\operatorname{Aut} S : S| = f$. However, by Lemma 3.2, $q^2 = 2^f \geqslant 2f$, and so

$$q^4 + 1 > q^2 q^2 = q^2 \cdot 2^f \geqslant q^2 \cdot 2f > f,$$

a contradiction.

Type G_2. If $S \cong G_2(q)$, then $\chi_2(1) = \frac{1}{2}q(q-1)(q^3-1)$ and $\chi_3(1) = \frac{1}{2}q(q+1)(q^3+1)$. In this case, $|\operatorname{Aut} S : S| = fg$, where $q = p^f$, $g = 2$ if $p = 3$, and $g = 1$ if $p \neq 3$. We assume $a\chi_2(1) = b\chi_3(1)$, and hence $a(q-1)(q^3-1) = b(q+1)(q^3+1)$.

If q is even, then $((q-1)(q^3-1), (q+1)(q^3+1)) = 1$, and so $(q-1)(q^3-1) \mid b$ and $(q+1)(q^3+1) \mid a$. Thus $(q^2-1)(q^6-1)$ divides $|\operatorname{Aut} S : S| = f$. However, by Lemma 3.2, $q = p^f \geqslant 2f$, and so $(q^2-1)(q^6-1) > q > f$, a contradiction.

If q is odd, then $((q-1)(q^3-1),(q+1)(q^3+1)) = 4$, and so in particular $\frac{1}{4}(q+1)(q^3+1) \mid a$. Thus $\frac{1}{4}(q+1)(q^3+1)$ divides $|\operatorname{Aut} S : S|$, hence also divides $2f$. However, $q > 2$ and $q = p^f \geqslant 2f$ by Lemma 3.2, hence

$$\frac{1}{4}(q+1)(q^3+1) > \frac{1}{4}q^4 \geqslant \frac{1}{4}q^3 \cdot 2f > 2f,$$

a contradiction.

Type $^2G_2$. Let $S \cong {}^2G_2(q^2)$, where $q^2 = 3^{2m+1}$ with $m \geqslant 1$. Let $\Phi'_{12} = q^2 - \sqrt{3}q + 1$ and $\Phi''_{12} = q^2 + \sqrt{3}q + 1$, so that $\Phi'_{12}\Phi''_{12} = \Phi_{12} = q^4 - q^2 + 1$. We have $\chi_2(1) = \frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi'_{12}$ and $\chi_3(1) = \frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi''_{12}$. In this case, $|\operatorname{Aut} S : S| = f$, where $q^2 = 3^f$; that is, $f = 2m+1$. We assume $a\chi_2(1) = b\chi_3(1)$, and hence $a\Phi'_{12} = b\Phi''_{12}$.

Since $(\Phi'_{12}, \Phi''_{12}) = 1$, we have $\Phi'_{12} \mid b$ and $\Phi''_{12} \mid a$. Thus $\Phi'_{12}\Phi''_{12} = \Phi_{12} = q^4 - q^2 + 1$ divides $|\operatorname{Aut} S : S| = f$. However, by Lemma 3.2, $q^2 = 3^f \geqslant 2f$, and so

$$q^4 - q^2 + 1 > (q^2-1)q^2 > q^2 > f,$$

a contradiction. □

4 Subgroups of $\operatorname{Aut}(\mathrm{PSL}_2(q))$

We have shown that if no prime divides three degrees of G, where $S \leqslant G \leqslant \operatorname{Aut} S$ for a simple group S, then $S \cong \mathrm{PSL}_2(q)$ for some prime power q. In order to complete the proof of Theorem 1, we must determine the groups G with $\mathrm{PSL}_2(q) \leqslant G \leqslant \operatorname{Aut} \mathrm{PSL}_2(q)$ such that no prime divides three degrees. This is accomplished in Lemmas 4.6, 4.7, 4.8, 4.9, and 4.10.

We first require some preliminary results on the actions of automorphisms of $\mathrm{PSL}_2(q)$.

4.1 Actions of Automorphisms on Irreducible Characters

We now consider the case $S \cong \mathrm{PSL}_2(q)$, $q = p^f \geqslant 5$ for a prime p, and $S \leqslant G \leqslant \operatorname{Aut} S$. We will require some detailed information about the actions of automorphisms of $\mathrm{PSL}_2(q)$ on its irreducible characters. We will use the notation of [5] for the conjugacy classes and characters of $\mathrm{SL}_2(q)$.

As is well-known, the outer automorphism group of $\mathrm{SL}_2(q)$ is of order $(2, q-1) \cdot f$, and is generated by a diagonal automorphism δ of order $(2, q-1)$ and a field automorphism φ of order f. (See [4], for example.) Moreover, δ and φ commute modulo inner automorphisms. If q is odd, then the center Z of $\mathrm{SL}_2(q)$ is of order 2 and so is fixed elementwise by both δ and φ. Hence δ and φ induce automorphisms of the same orders on $\mathrm{PSL}_2(q) = \mathrm{SL}_2(q)/Z$. We will use the same notation for the automorphisms of both $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$. An irreducible character of $\mathrm{SL}_2(q)$ is an irreducible character of $\mathrm{PSL}_2(q)$ if and only if Z is contained in its kernel, and in this case it is fixed by an automorphism of $\mathrm{SL}_2(q)$ if and only if it is fixed by the automorphism induced on $\mathrm{PSL}_2(q)$.

As in [5], denote by a an element of order $q-1$ and by b an element of order $q+1$ in $\mathrm{SL}_2(q)$. If q is odd, let z denote the element of order 2 in the center of $\mathrm{SL}_2(q)$. We denote by c an element of order q and, if q is odd, d denotes an element of order q not conjugate to c. The set

$$\{1, z, c, d, zc, zd, a^\ell, b^m \mid 1 \leqslant \ell \leqslant \frac{q-1}{2} - 1,\ 1 \leqslant m \leqslant \frac{q+1}{2} - 1\}$$

is a complete set of conjugacy class representatives for $\mathrm{SL}_2(q)$ for odd q and

$$\{1, c, a^\ell, b^m \mid 1 \leqslant \ell \leqslant \frac{q}{2} - 1,\ 1 \leqslant m \leqslant \frac{q}{2}\}$$

is a complete set of class representatives for even q. It is straightforward to verify the following result.

Lemma 4.1. *Assume the notation introduced above.*
If q is odd then

i. *δ fixes the classes of 1, z, a^ℓ, and b^m;*

ii. *δ interchanges the classes of c and d and interchanges the classes of zc and zd.*

For all $q = p^f$, $f \geqslant 2$, k a positive divisor of f,

iii. *φ^k fixes the classes of 1, z, c, zc, d, and zd if q is odd and fixes the classes of 1 and c if q is even;*

iv. *φ^k sends a^ℓ to the class of a^r, where $\ell p^k \equiv \pm r (\text{mod } q-1)$;*

v. *φ^k sends b^m to the class of b^s, where $mp^k \equiv \pm s (\text{mod } q+1)$.*

In all cases, $\mathrm{SL}_2(q)$ has the principal character of degree 1 and the Steinberg character, St, of degree q. Both of these are irreducible characters of $S = \mathrm{PSL}_2(q)$, invariant under δ and φ, and they extend to irreducible characters of $\mathrm{Aut}\, S$ (see [15]).

Also in all cases, $\mathrm{SL}_2(q)$ has irreducible characters χ_i of degree $q+1$ for $1 \leqslant i \leqslant [q/2]-1$, and θ_j of degree $q-1$ for $1 \leqslant j \leqslant [q/2]$. (Here, $[x]$ denotes the greatest integer less than or equal to the number x.) If q is odd, then χ_i, θ_j is an irreducible character of $S = \mathrm{PSL}_2(q)$ if and only if i, j, respectively, is even.

If q is odd, let $\epsilon = (-1)^{(q-1)/2}$. In this case, $\mathrm{SL}_2(q)$ has two irreducible characters of degree $(q+1)/2$ and two of degree $(q-1)/2$. Only the two characters of degree $(q+\epsilon)/2$ are irreducible characters of $S = \mathrm{PSL}_2(q)$, and we denote these by μ_1, μ_2.

Lemma 4.2. *If q is odd, then φ fixes the characters μ_1, μ_2, while $\mu_1^\delta = \mu_2$ and $\mu_2^\delta = \mu_1$; thus for $S \leqslant G \leqslant \mathrm{Aut}\, S$, the inertia group of μ_1 and μ_2 in G is $I_G(\mu_i) = G \cap S\langle\varphi\rangle$.*

Proof. This follows from Lemma 4.1 and the character table of $\mathrm{SL}_2(q)$ in [5]. □

Lemma 4.3. *If q is odd, then δ fixes χ_i and θ_j for all i, j. For all $q = p^f$, $f \geqslant 2$, and k a positive divisor of f,*

i. *φ^k fixes χ_i if and only if $p^f - 1 \mid (p^k-1)i$ or $p^f - 1 \mid (p^k+1)i$;*

ii. *φ^k fixes θ_j if and only if $p^f + 1 \mid (p^k+1)j$ or $p^f + 1 \mid (p^k-1)j$.*

Proof. By the character table of $\mathrm{SL}_2(q)$ in [5] and Lemma 4.1, all χ_i have the same value on the classes that are not fixed by δ, and similarly for all θ_j. Hence all χ_i and θ_j are fixed by δ.

Among classes not fixed by φ^k, the values of the χ_i differ only on the classes of a^ℓ. Denoting by ρ a complex primitive $(q-1)^{th}$ root of unity, we have $\chi_i(a^\ell) = \rho^{i\ell} + \rho^{-i\ell}$. Now $\chi_i^{\varphi^k} = \chi_i$ if and only if $\chi_i((a^{\varphi^{-k}})^\ell) = \chi_i(a^\ell)$ for $1 \leqslant \ell \leqslant [q/2]-1$. Hence χ_i is fixed by φ^k if and only if

$$\rho^{ip^k\ell} + \rho^{-ip^k\ell} = \rho^{i\ell} + \rho^{-i\ell}$$

for $1 \leqslant \ell \leqslant [q/2]-1$. It is easy to check that this holds if and only if

$$ip^k\ell \equiv \pm i\ell (\text{mod } q-1)$$

for all ℓ. This holds for all ℓ if and only if it holds for $\ell = 1$. Hence we have χ_i is fixed by φ^k if and only if $ip^k \equiv \pm i (\text{mod } q-1)$, and (i) follows.

Of the classes not fixed by φ^k, the values of the θ_j differ only on the classes of b^m. Denoting by σ a complex primitive $(q+1)^{th}$ root of unity, we have $\theta_j(b^m) = -(\sigma^{jm} + \sigma^{-jm})$. Now $\theta_j^{\varphi^k} = \theta_j$ if and only if $\theta_j((b^{\varphi^{-k}})^m) = \theta_j(b^m)$ for $1 \leqslant m \leqslant [q/2]$. Hence θ_j is fixed by φ^k if and only if

$$\sigma^{jp^k m} + \sigma^{-jp^k m} = \sigma^{jm} + \sigma^{-jm}$$

for $1 \leqslant m \leqslant [q/2]$. It is easy to check that this holds if and only if

$$jp^k m \equiv \pm jm (\text{mod } q+1)$$

for all m. This holds for all m if and only if it holds for $m = 1$. Hence we have θ_j is fixed by φ^k if and only if $jp^k \equiv \pm j (\text{mod } q+1)$, and (ii) follows. □

For the next lemma, observe that $|\operatorname{PGL}_2(q) : \operatorname{PSL}_2(q)| = (2, q-1)$ and, if q is odd, then $\operatorname{PGL}_2(q) = \operatorname{PSL}_2(q)\langle\delta\rangle = S\langle\delta\rangle$. Hence for $S \leqslant G \leqslant \operatorname{Aut}(S)$, if q is even, then $G \cap \operatorname{PGL}_2(q) = S$, and if q is odd, then $G \cap \operatorname{PGL}_2(q) = \operatorname{PGL}_2(q)$ if $\delta \in G$ and $G \cap \operatorname{PGL}_2(q) = S$ otherwise.

Lemma 4.4. *Let $q = p^f \geqslant 5$, where p is prime, $f \geqslant 2$, and $q \neq 9$. If $S \leqslant G \leqslant \operatorname{Aut} S$, then there exist characters $\chi, \theta \in \operatorname{Irr}(S)$ of degree $q+1$, $q-1$, respectively, whose inertia groups in G are $I_G(\chi) = I_G(\theta) = G \cap S\langle\delta\rangle = G \cap \operatorname{PGL}_2(q)$.*

Proof. First assume $q = 2^f$ is even, so that $f \geqslant 3$. Observe that if $1 \leqslant k < f$, then

$$2^f - 2^k = 2^k(2^{f-k} - 1) > 2$$

since $k \geqslant 1$ and $f \geqslant 3$. Therefore $2^f - 1 > 2^k + 1$.

In this case, we have $\operatorname{PGL}_2(q) = \operatorname{PSL}_2(q)$, so $S \leqslant G \leqslant S\langle\varphi\rangle$, and S has irreducible characters χ_1, θ_1 of degree $q+1$, $q-1$, respectively. If $k < f$ is a positive divisor of f, then since $2^f - 1 > 2^k + 1$, neither condition (i) nor condition (ii) of Lemma 4.3 can hold with $i = 1$ or $j = 1$. It follows that S is the inertia group of both $\chi = \chi_1$ and $\theta = \theta_1$ in G.

Now let $q = p^f$ for an odd prime p and $f \geqslant 2$, with $q \neq 9$. Note that if $1 \leqslant k < f$ and $p^f - 1 \leqslant 2(p^k + 1)$, then $p^k(p^{f-k} - 2) \leqslant 3$. But $p \geqslant 3$, and so this implies $p = 3$, $k = 1$, and $f = 2$, contradicting $q = p^f \neq 9$. Hence we have that if $1 \leqslant k < f$, then $p^f - 1 > 2(p^k + 1)$.

Since $2 < (q-3)/2$, both χ_2 and θ_2 are irreducible characters of $\operatorname{PSL}_2(q)$, and both are fixed by δ. If $k < f$ is a positive divisor of f, then since $p^f - 1 > 2(p^k + 1)$, neither condition (i) nor condition (ii) of Lemma 4.3 can hold with $i = 2$ or $j = 2$. It follows that $G \cap S\langle\delta\rangle = G \cap \operatorname{PGL}_2(q)$ is the inertia group of both $\chi = \chi_2$ and $\theta = \theta_2$ in G. □

We note that in the case $q = 9$, $S \cong \operatorname{PSL}_2(9) \cong A_6$ and both δ and φ are of order 2. There is a character θ of S of degree $q - 1 = 8$ with $I_G(\theta) = G \cap \operatorname{PGL}_2(q)$ as in the lemma. However, there is only one irreducible character of S of degree $q + 1 = 10$, which is therefore invariant in $\operatorname{Aut} S$.

Lemma 4.5. *Let $q = p^f \geqslant 5$, where p is prime, $f \geqslant 2$, and $q \neq 9$. If $S \leqslant G \leqslant \operatorname{Aut} S$, then G has irreducible characters of degrees $(q+1)|G : G \cap \operatorname{PGL}_2(q)|$ and $(q-1)|G : G \cap \operatorname{PGL}_2(q)|$.*

Proof. Since $(G \cap \operatorname{PGL}_2(q))/S$ is cyclic (of order 1 or 2), the characters χ, θ of Lemma 4.4 extend to $\tilde{\chi}, \tilde{\theta} \in \operatorname{Irr}(G \cap \operatorname{PGL}_2(q))$. By Clifford's theorem, $\tilde{\chi}_2^G$, $\tilde{\theta}_2^G$ are irreducible characters of G of degree $(q+1)|G : G \cap \operatorname{PGL}_2(q)|$, $(q-1)|G : G \cap \operatorname{PGL}_2(q)|$, respectively. □

4.2 $\mathrm{Aut}(\mathrm{PSL}_2(q))$, q Odd

We now let $S = \mathrm{PSL}_2(q)$, $q = p^f \geqslant 5$, p an odd prime, and G a group with $S < G \leqslant \mathrm{Aut}\, S$. We first consider the cases where G is contained in either $S\langle\delta\rangle = \mathrm{PGL}_2(q)$ or $S\langle\varphi\rangle$.

Lemma 4.6. *No prime divides three degrees of* $\mathrm{PGL}_2(q)$.

Proof. By the character table of $G = S\langle\delta\rangle = \mathrm{PGL}_2(q)$ in [17], we have $\mathrm{cd}(G) = \{1, q-1, q, q+1\}$. As 1 and q are relative prime to the other degrees, no prime can divide three degrees. □

In the next results, we consider $S < G \leqslant S\langle\varphi\rangle$, so that $G \cap \mathrm{PGL}_2(q) = S$. In particular, $G = S\langle\varphi^\ell\rangle$ for some $\ell \mid f$ with $1 \leqslant \ell < f$.

Lemma 4.7. *Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$, $p \neq 3$ is an odd prime, and $f \geqslant 2$. If $S < G \leqslant S\langle\varphi\rangle$, then there is a prime that divides three degrees of G.*

Proof. Let $|G : S| = 2^a m > 1$, where $a \geqslant 0$ and $m \geqslant 1$ is odd. By Lemma 4.5, we know that G has characters of degrees $(q+1)|G : S|$ and $(q-1)|G : S|$. We consider the cases $m = 1$ and $m \neq 1$ separately.

Suppose first that $m = 1$, so that $|G : S| = 2^a > 1$ and f must be even. Let

$$i = \frac{p^f - 1}{p - 1} = p^{f-1} + p^{f-2} + \cdots + p + 1.$$

Since $p \geqslant 5$, we have that $i < (q-1)/2$, hence $i \leqslant (q-3)/2$, and so $\chi_i \in \mathrm{Irr}(\mathrm{SL}_2(q))$. Since p is odd and f is even, we have that i is even, and so $\chi_i \in \mathrm{Irr}(S)$. Observe that $p^f - 1 \mid (p-1)i$, hence χ_i is invariant under φ by Lemma 4.3. Therefore, χ_i is invariant in G and since G/S is cyclic, χ_i extends to an irreducible character $\tilde{\chi}_i$ of G of degree $q+1$. Hence G has degrees $q+1$, $(q+1)|G : S|$, and $(q-1)|G : S|$, which are distinct and even.

We now suppose $m \neq 1$. Let $k = f/m$, so that $f/k = m$ is odd and hence $p^k + 1 \mid p^f + 1$. Let $j = 2(p^f + 1)/(p^k + 1)$, an even integer. Since $p^k + 1 \geqslant 5$, we have that $j < (q+1)/2$. Therefore $j \leqslant (q-1)/2$, so that $\theta_j \in \mathrm{Irr}(\mathrm{SL}_2(q))$, and since j is even, $\theta_j \in \mathrm{Irr}(S)$. Observe that $p^f + 1 \mid (p^k + 1)j$, and hence θ_j is invariant under φ^k.

Since the order m of φ^k divides $|G : S|$, we have $\varphi^k \in G$ and so $S\langle\varphi^k\rangle \leqslant I \leqslant G$, where I is the stabilizer of θ_j in G. Thus θ_j extends to an irreducible character $\tilde{\theta}_j$ of I, and $\tilde{\theta}_j$ induces to an irreducible character of G of degree $(q-1)|G : I|$. Therefore, G has degrees $(q-1)|G : I|$, $(q-1)|G : S|$, and $(q+1)|G : S|$, which are all even and, since $m \neq 1$, are distinct. □

Lemma 4.8. *Let $S \cong \mathrm{PSL}_2(q)$, where $q = 3^f$ and $f \geqslant 2$. If $S < G \leqslant S\langle\varphi\rangle$, then no prime divides three degrees of G if and only if f is a prime and $f \neq 3$.*

Proof. We first consider the case where f is prime, hence $G = S\langle\varphi\rangle$ and $|G : S| = f$. If $f = 2$, then $S \cong \mathrm{PSL}_2(9) \cong A_6$. In this case, we have $G = S\langle\varphi\rangle \cong S_6$ and $\mathrm{cd}(G) = \{1, 5, 9, 10, 16\}$, and so no prime divides three degrees of G, as claimed. However, if $f = 3$, then $S \cong \mathrm{PSL}_2(27)$ and G has degrees $q = 3^3$, $(q-1)f = 26 \cdot 3$, and $(q+1)f = 28 \cdot 3$ that are divisible by 3, using Lemma 4.5 or the Atlas [4].

We may now assume that $f \geqslant 5$ is prime, $S \cong \mathrm{PSL}_2(3^f)$, and $G = S\langle\varphi\rangle$, so that $|G : S| = f$. Moreover, G/S is cyclic of prime order, hence each irreducible character of S either extends to an irreducible character of G, if invariant under φ, or induces to an irreducible character of G, if not invariant under φ.

We have

$$\mathrm{cd}(S) = \{1, 3^f - 1, 3^f, 3^f + 1, (3^f - 1)/2\}.$$

By [15], the Steinberg character of degree 3^f extends to G and by Lemma 4.2, the characters of degree $(3^f - 1)/2$ extend to G. Of course, the only character of degree 1 is the principal character, which also extends to G.

The characters of degree 3^f-1 are the characters θ_j, where j is even and $1 \leqslant j \leqslant (3^f-1)/2$. By Lemma 4.3, θ_j is invariant under φ if and only if $3^f+1 \mid (3^1+1)j$ or $3^f+1 \mid (3^1-1)j$. Hence if θ_j is invariant under φ, then $(3^f+1)/4 \mid j$. Since f is odd, $3^f+1 \equiv 4 \pmod 8$, so that $(3^f+1)/4$ is odd, and since j is even, we must have $(3^f+1)/2 \mid j$. This contradicts $1 \leqslant j \leqslant (3^f-1)/2$, hence θ_j is not invariant under φ and θ_j^G is an irreducible character of G of degree $(3^f-1)f$.

The characters of degree 3^f+1 are the characters χ_i, where i is even and $1 \leqslant i \leqslant (3^f-3)/2$. By Lemma 4.3, χ_i is invariant under φ if and only if $3^f-1 \mid (3^1-1)i$ or $3^f-1 \mid (3^1+1)i$. Hence if χ_i is invariant under φ, then $3^f-1 \mid 4i$. Since f is odd, $3^f-1 \equiv 2 \pmod 4$, so $(3^f-1)/2$ is odd and this implies $(3^f-1)/2 \mid i$. This contradicts $1 \leqslant i \leqslant (3^f-3)/2$, hence χ_i is not invariant under φ and χ_i^G is an irreducible character of G of degree $(3^f+1)f$.

We now have that

$$\mathrm{cd}(G) = \{1, 3^f, (3^f-1)f, (3^f+1)f, (3^f-1)/2\}.$$

Since $f \neq 3$, 3^f is relatively prime to all other degrees. Since f is odd, $(3^f-1)/2$ is odd and relatively prime to 3^f+1. By Fermat's theorem, $3^f \equiv 3 \pmod f$, hence $f \nmid 3^f-1$ and so $(3^f-1)/2$ and $(3^f+1)f$ are relatively prime. Therefore no prime divides three degrees of G.

Next, we consider the case where f is not prime. By Lemma 4.5, G has degrees $(3^f-1)|G:S|$ and $(3^f+1)|G:S|$ in any case.

Suppose first that f is even and let $i = (3^f-1)/4$. We have that $i \leqslant (3^f-3)/2$ and since f is even i is also even, hence $\chi_i \in \mathrm{Irr}(S)$. Observe that $3^f-1 \mid (3^1+1)i$, and so χ_i is invariant under φ by Lemma 4.3. Therefore, χ_i extends to an irreducible character of G of degree 3^f+1. Since $|G:S|>1$, the degrees 3^f+1, $(3^f-1)|G:S|$, and $(3^f+1)|G:S|$ are distinct and even.

Finally, we assume f is odd and is not prime. Let $|G:S|=m$, so that $1<m\leqslant f$ and $m \mid f$. There is an integer ℓ such that $1<\ell<f$ and $\ell \mid m$. Let $k=f/\ell$ and set $j = 2(3^f+1)/(3^k+1)$. Note that f is odd and $k \mid f$, so $(3^f+1)/(3^k+1)$ is an integer and j is even. Since $k>1$, $3^k+1>4$, so that $j<(3^f+1)/2$. Hence $j \leqslant (3^f-1)/2$ and so $\theta_j \in \mathrm{Irr}(S)$. Observe that $3^f+1 \mid (3^k+1)j$, hence θ_j is invariant under φ^k.

Since $S\langle\varphi\rangle/S$ is cyclic and the order of φ^k is $\ell>1$, which divides $|G:S|$, we have that $\varphi^k \in G$. Hence $S < S\langle\varphi^k\rangle \leqslant I \leqslant G$, where I is the stabilizer of θ_j in G. Thus θ_j extends to an irreducible character $\tilde{\theta}_j$ of I, and $\tilde{\theta}_j$ induces to an irreducible character of G of degree $(3^f-1)|G:I|$. Therefore, G has degrees $(3^f-1)|G:I|$, $(3^f-1)|G:S|$, and $(3^f+1)|G:S|$, which are all even and, since $S<I$, are distinct. □

Finally, we consider the case where $S < G \leqslant \mathrm{Aut}\, S$ and G is not contained in either of $S\langle\delta\rangle = \mathrm{PGL}_2(q)$ or $S\langle\varphi\rangle$. In particular, $|G : G\cap S\langle\varphi\rangle| = 2$ and $|G : G \cap \mathrm{PGL}_2(q)| > 1$.

Lemma 4.9. *Let $S \cong \mathrm{PSL}_2(q)$, where $q=p^f$, p is an odd prime, and $f \geqslant 2$. If $S<G\leqslant \mathrm{Aut}\, S$ but G is contained in neither of $S\langle\delta\rangle = \mathrm{PGL}_2(q)$ or $S\langle\varphi\rangle$, then there is a prime that divides three degrees of G, except in the case where $q=9$ and $G = S\langle\delta\varphi\rangle \cong M_{10}$ with $|G:S|=2$.*

Proof. Let $q=3^2$, so that $S \cong \mathrm{PSL}_2(9) \cong A_6$. Either $G = S\langle\delta\varphi\rangle \cong M_{10}$ or $G = \mathrm{Aut}\, S$. If $|G:S|=2$, so that $G\cong M_{10}$, then the Atlas [4] character table shows that $\mathrm{cd}(G) = \{1,9,10,16\}$, and so no prime divides three degrees of G, as claimed. However, if $G=\mathrm{Aut}\, S$, it is not difficult to determine that $\mathrm{cd}(G)=\{1,9,10,16,20\}$ and G has three distinct even degrees. We may now assume $q>3^2$.

By Lemma 4.2, the inertia group of μ_1 in G is $I_G(\mu_1) = G\cap S\langle\varphi\rangle$, which is of index 2 in G. As $I_G(\mu_1)/S$ is cyclic, μ_1 extends to an irreducible character $\tilde{\mu}_1$ of $I_G(\mu_1)$, and then $\tilde{\mu}_1^G$ is an irreducible character of G of degree $|G:I_G(\mu_1)|\mu_1(1) = 2(q+\epsilon)/2 = q+\epsilon$. By Lemma 4.5, G also has characters of degrees $(q+1)|G:G\cap\mathrm{PGL}_2(q)|$ and $(q-1)|G:G\cap\mathrm{PGL}_2(q)|$. These three degrees are even and, since $|G:G\cap\mathrm{PGL}_2(q)|>1$, they are distinct. □

4.3 $\mathrm{Aut}(\mathrm{PSL}_2(2^f))$

We now consider the case $S \cong \mathrm{SL}_2(q) \cong \mathrm{PSL}_2(q) \cong \mathrm{PGL}_2(q)$, where $q = 2^f$, $f \geqslant 3$. (The case $q = 4$ is finished, as $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$.) We have $\mathrm{Aut}\, S = S\langle\varphi\rangle$ and $|\mathrm{Aut}\, S : S| = f$.

Lemma 4.10. *Let $S \cong \mathrm{SL}_2(q)$, where $q = 2^f$ and $f \geqslant 3$. If $S < G \leqslant \mathrm{Aut}\, S$, then no prime divides three degrees of G if and only if $|G : S| = f = 3$ or $|G : S|$ is an odd prime that divides neither of $q-1$, $q+1$.*

Proof. Suppose first that $|G : S|$ is even. By Lemma 4.5, G has degrees $(q-1)|G : S|$ and $(q+1)|G : S|$. The Steinberg character of S extends to a character of G of degree q, and so these three degrees are even and distinct.

We may now assume $|G : S| = m > 1$ is odd. Suppose m is not prime, and let ℓ be a divisor of m with $1 < \ell < m$. Let k be one of f/m, $f/(m/\ell)$, or f. In each case, $f/k = |S\langle\varphi^k\rangle : S|$ divides $m = |G : S|$, so that $S \leqslant S\langle\varphi^k\rangle \leqslant G$. It also follows that f/k is odd and so $2^k + 1 \mid 2^f + 1$. Let $j = (2^f+1)/(2^k+1)$, so that $j < (2^f+1)/2$, and so $j \leqslant 2^f/2 = q/2$. We therefore have that θ_j is an irreducible character of S of degree $q - 1$.

Observe that $2^f + 1 \mid (2^k+1)j$, hence by Lemma 4.3, θ_j is invariant under φ^k and we have $S \leqslant S\langle\varphi^k\rangle \leqslant I \leqslant G$, where I is the stabilizer of θ_j in G. If $S\langle\varphi^k\rangle < I$, then $I = S\langle\varphi^t\rangle$ for some divisor t of k with $1 \leqslant t < k$. Hence θ_j is invariant under φ^t and, by Lemma 4.3, $2^f + 1 \mid (2^t - 1)j$ or $2^f + 1 \mid (2^t + 1)j$. This implies $2^k + 1$ divides $2^t - 1$ or $2^t + 1$. However, this contradicts $k > t$, and so $I = S\langle\varphi^k\rangle$.

Since I/S is cyclic, θ_j extends to an irreducible character $\tilde\theta_j$ of I, and $\tilde\theta_j^G$ is an irreducible character of G of degree $(q-1)|G : I|$. Recall that $k = f/m$, $f/(m/\ell)$, or f, and $|I : S| = f/k$. Hence $|G : I| = m/(f/k)$ is 1, ℓ, or m, respectively. Therefore, G has degrees $q-1$, $(q-1)\ell$, and $(q-1)m$, which are distinct and divisible by a prime dividing $q - 1$.

Finally, suppose $|G : S| = m$ is an odd prime. We have $G = S\langle\varphi^\ell\rangle$, where $\ell = f/m$. In particular, G/S is cyclic of prime order, hence each irreducible character of S either extends to an irreducible character of G, if invariant under φ^ℓ, or induces to an irreducible character of G, if not invariant under φ^ℓ.

We have

$$\mathrm{cd}(S) = \{1, 2^f - 1, 2^f, 2^f + 1\}.$$

The characters of degree 1, 2^f are the principal character and the Steinberg character, respectively, which are both invariant under φ and so extend to G. By Lemma 4.5, we know that G has degrees $(2^f - 1)m$ and $(2^f + 1)m$. Hence in order to determine $\mathrm{cd}(G)$, it remains only to determine if there are characters of S of degree $2^f - 1$ or $2^f + 1$ invariant under φ^ℓ.

The characters of degree $2^f - 1$ are the characters θ_j, where $1 \leqslant j \leqslant q/2 = 2^f/2$. Since $f/\ell = m$ is odd, $2^\ell + 1 \mid 2^f + 1$. Let $j = (2^f+1)/(2^\ell+1)$, so that $j < 2^f/2 = q/2$ as above and $\theta_j \in \mathrm{Irr}(S)$. Since $2^f + 1 \mid (2^\ell + 1)j$, θ_j is invariant under φ^ℓ by Lemma 4.3 and θ_j extends to an irreducible character of G degree $2^f - 1$.

The characters of degree $2^f + 1$ are the characters χ_i, where $1 \leqslant i \leqslant (q/2) - 1 = (2^f - 2)/2$. By Lemma 4.3, χ_i is invariant under φ^ℓ if and only if $2^f - 1 \mid (2^\ell - 1)i$ or $2^f - 1 \mid (2^\ell + 1)i$.

If $f = m$ is an odd prime, so that $\ell = 1$ and $G = S\langle\varphi\rangle$, then $2^\ell - 1 = 1$ and $2^\ell + 1 = 3$. Since f is odd, $3 \nmid 2^f - 1$, and so χ_i is invariant in G if and only if $2^f - 1 \mid i$. Since $1 \leqslant i \leqslant (2^f/2) - 1$, this is not possible, and so no character of S of degree $2^f + 1$ is invariant in G. Hence if $f = m$ is an odd prime, we have

$$\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, (2^f - 1)f, (2^f + 1)f\}.$$

By Fermat's theorem, since f is an odd prime, $2^f \equiv 2 \pmod f$, and so $2^f - 1 \equiv 1 \pmod f$. Therefore, $f \nmid 2^f - 1$ and no prime divides three degrees of G in this case. If $f > 3$, then $f \nmid 2^f + 1$, and so we also have that f divides neither of $q - 1$, $q + 1$.

If m is an odd prime with $m < f$, so that $\ell > 1$, let $i = (2^f - 1)/(2^\ell - 1)$. Since $\ell \geqslant 2$, we have $i < (2^f - 1)/2$, hence $i \leqslant (2^f - 2)/2 = (q/2) - 1$ and $\chi_i \in \mathrm{Irr}(S)$. We have $2^f - 1 \mid (2^\ell - 1)i$, and so χ_i is invariant under φ^ℓ. Therefore χ_i extends to a character of G of degree $2^f + 1$. Hence if m is an odd prime with $m < f$, we have

$$\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, 2^f + 1, (2^f - 1)m, (2^f + 1)m\},$$

and so there is a prime dividing three degrees of G if and only if $m \mid 2^f - 1$ or $m \mid 2^f + 1$. □

This completes the proof of Theorem 1.

We observe that it is possible for an odd prime divisor m of f to also divide either $2^f - 1$ or $2^f + 1$, and thus divide three degrees of G when $S < G < \mathrm{Aut}\, S$ and $|G : S| = m$. In particular, let $\ell > 1$ be an integer and m any prime divisor of either $2^\ell - 1$ or $2^\ell + 1$. If $f = m\ell$, then since $2^m \equiv 2 \pmod{m}$, we have $2^f \equiv 2^\ell \pmod{m}$, and so m is a divisor of $2^f - 1$ or $2^f + 1$, respectively.

5 Nonsolvable Groups

5.1 Preliminary Results

We now shift to general nonsolvable groups. We first consider nonabelian minimal normal subgroups of a group where no prime divides three character degrees.

Lemma 5.1. *Let G be a group where no prime divides three character degrees. If N is a nonabelian minimal normal subgroup of G, then N is simple.*

Proof. There exists a nonabelian simple group S so that $N = S_1 \times \cdots \times S_t$, where t is a positive integer and $S_i \cong S$ for $i = 1, \ldots, t$. We know that G acts transitively on $\{S_1, \ldots, S_t\}$. We assume that $t > 1$, and work for a contradiction.

We know from [2] that there exists $\sigma \in \mathrm{Irr}(S)$ so that σ extends to $\mathrm{Aut}(S)$ with $\sigma(1) \geqslant 4$. Let $\gamma^* = \sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$. By [2], γ^* extends to G, and so, $\gamma^*(1) = \sigma(1)^t \in \mathrm{cd}(G)$. Let $\tau_1 = \sigma \times 1 \times \cdots \times 1 \in \mathrm{Irr}(N)$. The stabilizer of τ_1 in G will be the same as the normalizer of S_1. Now, τ_1 will extend to its stabilizer. It follows that $\tau_1(1)t = \sigma(1)t \in \mathrm{cd}(G)$. It is easy to see that $t < \sigma(1)^{t-1}$, so $\sigma(1)t < \sigma(1)^t$.

We can find $\delta \in \mathrm{Irr}(S)$ so that $\delta(1)$ is not 1 and is relatively prime to $\sigma(1)$ (see [2]). Let $\tau_2 = \sigma \times \delta \times 1 \times \cdots \times 1 \in \mathrm{Irr}(N)$. The stabilizer of τ_2 will be contained in the stabilizer of τ_1. It follows that $\mathrm{cd}(G)$ contains a degree a that is divisible by $\sigma(1)\delta(1)t$, and that a is neither $\sigma(1)^t$ nor $\sigma(1)t$. Let p be a prime divisor of $\sigma(1)$. We see that $\sigma(1)^t$, $\sigma(1)t$, and a are three degrees in $\mathrm{cd}(G)$ that are divisible by p, a contradiction. We conclude that $t = 1$. □

We now show that the centralizer of such a minimal normal subgroup will be central in G.

Lemma 5.2. *Let N be a normal subgroup of G such that N is nonabelian simple group. If no prime divides three degrees in $\mathrm{cd}(G)$, then $\mathbf{C}_G(N)$ is central in G.*

Proof. Let $C = \mathbf{C}_G(N)$ and denote $M = N \times C$. Observe that M is normal in G. Also, G/C is isomorphic to a subgroup of $\mathrm{Aut}(N)$, and G/M is isomorphic to a subgroup of $\mathrm{Out}(N)$. In particular, G/C is almost simple. By Theorem 1, we know that $N \cong \mathrm{PSL}_2(q)$ and either $G = M$ or $|G : M|$ is prime.

Fix $\sigma, \nu \in \mathrm{Irr}(N)$ with $\sigma(1) = q$ and $\nu(1) = q - 1$. Consider $\tau \in \mathrm{Irr}(C)$ with stabilizer T in G. Note that either $T = G$ or $T = M$. Observe that $1_N \times \tau$ and $\sigma \times \tau$ both have stabilizer T in G. Also, $\nu \times \tau$ has stabilizer S where $M \leqslant S \leqslant T$. Note that either $|G : T| = |G : S|$ or $|G : T| = 1$ and $|G : S| = |G : M|$. In either case, we have $|G : S|\nu(1) \neq |G : T|\sigma(1)$ since $q - 1$ does not divide q.

Applying Clifford's theorem, $|G : T|\tau(1)$, $|G : S|\nu(1)\tau(1)$, and $|G : T|\sigma(1)\tau(1)$ all lie in $\mathrm{cd}(G)$. If $\tau(1) > 1$, then this yields three degrees divisible by any prime divisor of $\tau(1)$. Thus, every character in $\mathrm{Irr}(C)$ is linear, and C is abelian. If $|G:T|>1$, then $|G:S|=|G:T|$, and the three degrees are $|G:T|, |G:T|\nu(1), |G:T|\sigma(1)$. We now have three degrees divisible by the prime divisors of $|G:T|$. We deduce that every irreducible character of C is G-invariant. Since also every irreducible character of C is linear, we conclude that C is central in G. □

We can apply these two results to show that nonabelian chief factors are simple, and that G has at most one such nonabelian simple factor in its derived series.

Corollary 5.3. *Suppose $M<N$ are normal subgroups of G, so that N/M is a nonabelian chief factor for G. If no prime divides three degrees in $\mathrm{cd}(G)$, then N/M is simple, G/N is abelian, and M is solvable.*

Proof. Observe that G/M is nonsolvable, no prime divides three degrees in $\mathrm{cd}(G/M)$, and N/M is a minimal normal subgroup of G/M. The fact that N/M is simple comes from Lemma 5.1. The fact that G/N is abelian follows from Theorem 1.

We work to show that M is solvable. If M is not solvable, then we can find $L<K\leqslant M$ so that L and K are normal in G and K/L is a nonabelian chief factor for G. By Lemma 5.1, K/L is a simple group. Let $C/L=\mathbf{C}_{G/L}(K/L)$. We know that G/CK is abelian, and so $C/L\cong CK/K$ is not solvable. This contradicts Lemma 5.2, which implies that C/L is central and hence abelian. Therefore, M is solvable. □

Corollary 5.4. *If G is a nonsolvable group so that no prime divides three degrees in $\mathrm{cd}(G)$, then there exist normal subgroups $L<K$ in G so that K/L is simple, L is solvable, G/K is abelian, and $G/L\leqslant \mathrm{Aut}(K/L)$, so G/L is almost simple.*

Proof. Let L be maximal among normal subgroups of G so that G/L is not solvable. Let K be normal in G so that K/L is a chief factor for G. Since G/K is solvable, we conclude that K/L is nonabelian. By Lemma 5.3, K/L is simple, G/K is abelian, and L is solvable. Let $C/L=\mathbf{C}_{G/L}(K/L)$. By Lemma 5.2, C/L is abelian, which implies that C is solvable. Hence, $C=L$, and so, $G/L\leqslant \mathrm{Aut}(K/L)$. □

5.2 Almost Simple Sections

The previous result shows that if G is a nonsolvable group and no prime divides three degrees of G, then G has a normal subgroup L so that G/L is an almost simple group with no prime dividing three degrees. We now use the classification of these groups in Theorem 1. With this in mind, we make the following hypothesis.

Hypothesis (*). *No prime divides three degrees in $\mathrm{cd}(G)$; $L<K$ are normal subgroups of G so that $K/L\cong \mathrm{PSL}_2(q)$ and $G/L\leqslant \mathrm{Aut}(K/L)$ (i.e., G/L is almost simple). Fix $\tau\in\mathrm{Irr}(L)$ and let T be the stabilizer of τ in G.*

Throughout, we will make use of Dickson's list of the subgroups of $\mathrm{PSL}_2(q)$, which can be found as Hauptsatz II.8.27 of [6]. We also use the fact that the Schur multiplier of $\mathrm{PSL}_2(q)$ is trivial unless $q=4$ or q is odd, in which case it is of order 2 if $q\neq 9$ and of order 6 if $q=9$. We frequently use Clifford's theorem, which can be found as Theorem 6.11 of [8], and Gallagher's theorem, which is Corollary 6.17 of [8]. If H is a subgroup of G and $\tau\in\mathrm{Irr}(H)$, we denote by $\mathrm{Irr}(G\mid\tau)$ the set of irreducible characters of G lying over τ and define $\mathrm{cd}(G\mid\tau)=\{\chi(1)\mid\chi\in\mathrm{Irr}(G\mid\tau)\}$.

We begin with the following simple observation.

Lemma 5.5. *Assume Hypothesis (*). If $T<G$, then $|\mathrm{cd}(G\mid\tau)|=|\mathrm{cd}(T\mid\tau)|\leqslant 2$.*

Proof. The conclusion follows from the fact that $|G:T|$ divides every degree in $\mathrm{cd}(G\mid\tau)$ and the observation from Clifford theory that $|\mathrm{cd}(G\mid\tau)|=|\mathrm{cd}(T\mid\tau)|$. □

We next show that τ is not K-invariant when $T < G$.

Lemma 5.6. *Assume Hypothesis (*). If $T < G$, then $K \nsubseteq T$.*

Proof. Suppose $K \subseteq T$, so that τ is K-invariant. If τ extends to K, then either $|\mathrm{cd}(T \mid \tau)| = 4$ if q is even or $q = 5$, or $|\mathrm{cd}(T \mid \tau)| = 5$ if $q \geqslant 7$ is odd, and this violates Lemma 5.5. Thus, we may assume that τ does not extend to K. This implies that K/L has a nontrivial Schur multiplier, and so either $q = 4$ or q is odd. As long as $q \neq 9$ or $q = 9$ and τ corresponds to the character of order 2, we see that $|\mathrm{cd}(T \mid \tau)| = 3$, a contradiction of Lemma 5.5. If $q = 9$ and τ corresponds to a character of order 3, we deduce that $|\mathrm{cd}(T \mid \tau)| = 4$, again contradicting Lemma 5.5. The final possibility is that $q = 9$ and τ corresponds to a character of order 6. In this case, $\mathrm{cd}(T \mid \tau) = \{6\tau(1), 12\tau(1)\}$, and so $\mathrm{cd}(G \mid \tau) = \{12\tau(1), 24\tau(1)\}$. Since $10, 16 \in \mathrm{cd}(G/L)$, this yields a contradiction. □

Next, we consider possibilities for the stabilizer in K (modulo L). First, we show that this cannot be A_5.

Lemma 5.7. *Assume Hypothesis (*). If $q \geqslant 7$, then $(T \cap K)/L$ is not A_5.*

Proof. Suppose that $(T \cap K)/L$ is A_5. Since $q > 5$, we know that $|K : T \cap K| > 1$, and so $|G : T| > 1$. If τ extends to $T \cap K$, then we have $3\tau(1), 4\tau(1), 5\tau(1) \in \mathrm{cd}(T \cap K \mid \tau)$. Notice that $|T : T \cap K|$ divides $|G : K|$, which, by Theorem 1, divides a prime. Thus we get three distinct degrees in $\mathrm{cd}(T \mid \tau)$, contradicting Lemma 5.5.

If τ does not extend to $T \cap K$, then $4\tau(1), 6\tau(1) \in \mathrm{cd}(T \cap K \mid \tau)$. Since $|T : T \cap K|$ divides a prime, we get two distinct degrees $a, b \in \mathrm{cd}(T \mid \tau)$ that are bigger than 1. This yields $a|G : T|$ and $b|G : T|$ are in $\mathrm{cd}(G \mid \tau)$. Since $|(T \cap K)/L| = |A_5| = 60$ and q is a power of prime p with $q > 5$, we conclude that p divides $|K : T \cap K|$, and so p divides $|G : T|$. Also, we know that $|G : T| > q$ unless $q = 11$, by Dickson's classification. It follows that $a|G : T|$ and $b|G : T|$ are greater than q, and so q, $a|G : T|$, and $b|G : T|$ will be distinct degrees in $\mathrm{cd}(G)$ divisible by p, a contradiction. □

We also show that the stabilizer in K (modulo L) cannot be $\mathrm{PSL}_2(r)$ or $\mathrm{PGL}_2(r)$, where r properly divides q.

Lemma 5.8. *Assume Hypothesis (*). Then $(T \cap K)/L$ is neither $\mathrm{PSL}_2(r)$ nor $\mathrm{PGL}_2(r)$, where $r > 3$ is a proper divisor of q.*

Proof. Suppose that $(T \cap K)/L$ is $\mathrm{PSL}_2(r)$ or $\mathrm{PGL}_2(r)$ with $r > 3$. If $r = 4$ or 5, then $(T \cap K)/L$ is A_5, and we are done by Lemma 5.7. Thus, we may assume $r > 5$. Let p be the prime dividing q and r. Observe that $|K : T \cap K|$ is divisible by p but is not a power of p. It follows that $|G : T|$ is divisible by p but is not a power of p.

Assume one of the following conditions: (1) τ extends to $T \cap K$, (2) τ does not extend to $T \cap K$ and $r \neq 9$, or (3) τ does not extend to $T \cap K$, $r = 9$, and τ corresponds to the character of order 2. We have that $\tau(1)(r-1), \tau(1)(r+1) \in \mathrm{cd}(T \cap K \mid \tau)$. Since $|T : T \cap K|$ divides a prime, we get distinct degrees $a, b \in \mathrm{cd}(T \mid \tau)$ that are greater than 1, and so $a|G : T|$ and $b|G : T|$ are in $\mathrm{cd}(G \mid \tau)$. Along with q, these are three degrees in $\mathrm{cd}(G)$ that are divisible by p.

Now, suppose τ does not extend to $T \cap K$ and $r = 9$. Suppose first that τ corresponds to a character of order 3. Notice that this implies $(T \cap K)/L \cong \mathrm{PSL}_2(9)$. It follows that

$$\mathrm{cd}(T \cap K \mid \tau) = \{3\tau(1), 6\tau(1), 9\tau(1), 15\tau(1)\}.$$

Since $|T : T \cap K|$ divides a prime, this gives at least three degrees in $\mathrm{cd}(T \mid \tau)$, violating Lemma 5.5.

Finally, suppose τ corresponds to a character of order 6. Again, $(T \cap K)/L \cong \mathrm{PSL}_2(9)$. By Lemma 5.6, $T \cap K \neq K$, so we have $q = 3^f$ where $f > 2$. Hence, $|T : T \cap K|$ is odd. We have $\mathrm{cd}(T \cap K \mid \tau) = \{6\tau(1), 12\tau(1)\}$. This implies $|\mathrm{cd}(T \mid \tau)| = 2$, and we get three degrees in $\mathrm{cd}(G)$ divisible by $p = 3$. □

5.3 When q is Odd

In this section, we consider the case where $\mathrm{PSL}_2(q)$ is involved in G and q is odd. We begin by showing that the characters of L are G-invariant. This contains the work common to all cases.

Lemma 5.9. *Assume Hypothesis (*). If $q \geqslant 7$ is odd, then $T = G$.*

Proof. We begin by noting that $\mathrm{cd}(G/L)$ must have two even degrees:

1. $q-1, q+1$ if G/L is either $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$,
2. $10, 16$ if $G/L \cong S_6$ or M_{10} (with $q = 9$),
3. $(3^f-1)f, (3^f+1)f$ if $G/L \cong \mathrm{PSL}_2(3^f) \rtimes Z_f$, where $f \neq 3$ is an odd prime (with $q = 3^f$).

We suppose that $T < G$. By Lemma 5.6, we know that $T \cap K < K$. Also, we know that $|T : T \cap K|$ divides a prime, so $T/(T \cap K)$ is cyclic. By Lemmas 5.7 and 5.8, we know that $(T \cap K)/L$ is not A_5, $\mathrm{PSL}_2(r)$, or $\mathrm{PGL}_2(r)$, where r divides q. From Dickson's list of subgroups, it follows that $(T \cap K)/L$, and hence T/L, is solvable.

If $|G : T|$ is even, then all degrees in $\mathrm{cd}(G \mid \tau)$ are even, and so $\mathrm{cd}(G \mid \tau) \subseteq \mathrm{cd}(G/L)$. By Dickson's list of subgroups, we know that either $|T : T \cap K| \geqslant q+1$ and $(T \cap K)/L$ is a Frobenius group, or else $|T : T \cap K| = q$ with $q = 5, 7, 11$, or $|T : T \cap K| = 6$ and $q = 9$. Since $|T : T \cap K|$ must divide the degrees in $\mathrm{cd}(G \mid \tau)$, we obtain $|T : T \cap K| = q+1$, and $|\mathrm{cd}(G \mid \tau)| = 1$ since in no case does $q+1$ divide both even degrees.

There is a normal subgroup A in $T \cap K$ with $|T \cap K : A| = (q-1)/2$ and A/L elementary abelian of order q. If τ does not extend to A, then there exists $b \in \mathrm{cd}(G \mid \tau)$ such that p divides b, where p is the prime dividing q. However, none of the even degrees in $\mathrm{cd}(G)$ is divisible by p, so this is a contradiction.

Thus, τ extends to A. Note that A/L is the Sylow p-subgroup of $(T \cap K)/L$. If R/L is a Sylow subgroup of $(T \cap K)/L$ for some prime other than p, then R/L is cyclic, and τ extends to R by Corollary 11.22 of [8]. It follows that τ extends to $T \cap K$ by Corollary 11.31 of [8]. In light of Gallagher's theorem and the fact that $|\mathrm{cd}(T \mid \tau)| = 1$, we conclude that T/L is abelian, but this is also a contradiction since $(T \cap K)/L$ is not abelian.

We now suppose that $|G : T|$ is odd. In particular, T/L contains a full Sylow 2-subgroup of G/L. By Dickson's list, we know that $|G : T| = |K : T \cap K| \geqslant q$, and so any even degree in $\mathrm{cd}(G \mid \tau)$ will be larger than $q+1$, thus $\mathrm{cd}(G \mid \tau)$ has no even degrees. It follows that $\mathrm{cd}(T \mid \tau)$ has no even degrees. By Theorem 12.9 of [13], this implies that T/L has an abelian Sylow 2-subgroup, and so G/L has an abelian Sylow 2-subgroup.

It follows that $G/L \cong \mathrm{PSL}_2(q)$ and T/L is either a Klein 4-group or $T/L \cong A_4$. This follows from the fact that $\mathrm{PGL}_2(q)$ has a nonabelian Sylow 2-subgroup, and $\mathrm{PSL}_2(q)$ has a nonabelian Sylow 2-subgroup if 8 divides $|\mathrm{PSL}_2(q)|$. Since $\mathrm{cd}(T \mid \tau)$ has no even degrees, τ must extend to the Sylow 2-subgroup of T/L. Also, τ will extend to the Sylow 3-subgroup of T/L since it is cyclic. By Corollary 11.22 of [8], τ extends to T. Notice that either T/L is a Klein 4-group and $|G : T|$ is the odd part of $(q-1)q(q+1)/2$, or $T/L \cong A_4$ and $|G : T|$ is the odd part of $(q-1)q(q+1)/6$ and $3 \in \mathrm{cd}(T \mid \tau)$. In any case, $\mathrm{cd}(G \mid \tau)$ has a degree a that is the odd part of $(q-1)q(q+1)/2$. Since $q \geqslant 7$, we see that $a > q+1$. In this case, we know that $(q+\epsilon)/2 \in \mathrm{cd}(G/L)$, where $\epsilon \in \{\pm 1\}$ is such that $(q+\epsilon)/2$ is odd. Now any prime divisor of $(q+\epsilon)/2$ will divide three degrees in $\mathrm{cd}(G)$, a contradiction. □

We now consider the case where $K/L \cong \mathrm{PSL}_2(q)$ and G/L is isomorphic to a subgroup of $\mathrm{Aut}\,\mathrm{PSL}_2(q)$, with $q \geqslant 7$ odd. The case where $q = 5$ will be considered in the next section since $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$.

Theorem 5.10. *Let G be a nonsolvable group such that no prime divides three degrees in $\mathrm{cd}(G)$ and the nonabelian chief factor of G is isomorphic to $\mathrm{PSL}_2(q)$, where $q \geqslant 7$ is odd. If $L < K$ are normal subgroups such that $K/L \cong \mathrm{PSL}_2(q)$ and $G/L \leqslant \mathrm{Aut}(K/L)$, then $L = Z(G)$ and $\mathrm{cd}(G) = \mathrm{cd}(G/L)$.*

Proof. Fix subgroups $L < K$ so that $K/L \cong \mathrm{PSL}_2(q)$, $q \geqslant 7$ odd. By Theorem 1, G/L is isomorphic to one of $\mathrm{PSL}_2(q)$, $\mathrm{PGL}_2(q)$, $\mathrm{PSL}_2(3^f) \rtimes Z_f$ for some prime $f > 3$, or (if $q = 3^2$) S_6 or M_{10}. As in the proof of Lemma 5.9, observe that there are two even degrees e_1, e_2 in $\mathrm{cd}(G/L)$, hence in $\mathrm{cd}(G)$, and so these must be the only even degrees of G.

Consider $\tau \in \mathrm{Irr}(L)$. We know by Lemma 5.9 that τ is G-invariant. If τ extends to G, then using Gallagher's theorem, we obtain $\tau(1)e_1, \tau(1)e_2 \in \mathrm{cd}(G)$. Since these are even, we conclude that $\tau(1) = 1$ and $\mathrm{cd}(G \mid \tau) = \mathrm{cd}(G/L)$.

We now assume $\tau \in \mathrm{Irr}(L)$ does not extend to G.

Case 1: $G/L \cong \mathrm{PSL}_2(q)$ or $G/L \cong \mathrm{PGL}_2(q)$.

Note that $q-1$ and $q+1$ are the only even degrees in $\mathrm{cd}(G)$. If $q \neq 9$ so that the Schur multiplier of G/L is of order 2, or $q = 9$ and τ corresponds to a character of order 2, we obtain $(q-1)\tau(1), (q+1)\tau(1) \in \mathrm{cd}(G \mid \tau)$. Since these are even, we determine that $\tau(1) = 1$. Also, if $G/L \cong \mathrm{PGL}_2(q)$, we have that $\mathrm{cd}(G \mid \tau) \subseteq \mathrm{cd}(G/L)$. If $G/L \cong \mathrm{PSL}_2(q)$, then since $\mathrm{SL}_2(q)$ is the representation group, we obtain $(q-\epsilon)/2 \in \mathrm{cd}(G \mid \tau)$, where $\epsilon \in \{\pm 1\}$ is such that $(q+\epsilon)/2$ is odd. Now, $(q-\epsilon)/2$ is a third even degree, so this cannot occur.

If $q = 9$ and τ corresponds to a character of order 3, then

$$\mathrm{cd}(G \mid \tau) = \{3\tau(1), 6\tau(1), 9\tau(1), 15\tau(1)\}.$$

We have four degrees divisible by 3, a contradiction. If τ corresponds to a character of order 6, then $\mathrm{cd}(G \mid \tau) = \{6\tau(1), 12\tau(1)\}$. In this case, the even degrees in $\mathrm{cd}(G/L)$ are 8 and 10, so we get a contradiction.

Case 2: $G/L \cong \mathrm{PSL}_2(3^f) \rtimes Z_f$, where $f > 3$ is prime.

We begin by observing that

$$\{(3^f-1)/2, 3^f, (3^f-1)f, (3^f+1)f\} \subseteq \mathrm{cd}(G/L) \subseteq \mathrm{cd}(G)$$

and $(3^f-1)f$ and $(3^f+1)f$ are the only even degrees in $\mathrm{cd}(G)$.

If τ extends to K, then by Gallagher's theorem, τ has a unique extension $\hat{\tau} \in \mathrm{Irr}(K)$. It follows that $\hat{\tau}$ is also G-invariant, and since G/K is cyclic, we see that $\hat{\tau}$, and hence τ, extends to G, and we are done as above.

If τ does not extend to K, we obtain two characters of degree $\tau(1)(3^f+1)/2$ in $\mathrm{Irr}(K \mid \tau)$. Since f is an odd prime, each of these two characters must be G-invariant, and hence they extend to G. This yields $\tau(1)(3^f+1)/2 \in \mathrm{cd}(G \mid \tau)$. As this degree must be $(3^f-1)f$ or $(3^f+1)f$, we deduce that $\tau(1) = 2f$. Also, $\mathrm{Irr}(K \mid \tau)$ contains characters of degree $\tau(1)(3^f+1)$, so $\mathrm{cd}(G \mid \tau)$ will contain a degree divisible by

$$\tau(1)(3^f+1) = (3^f+1)2f > (3^f+1)f,$$

a third even degree. Thus, this case cannot occur.

Case 3: $G/L \cong S_6$ or $G/L \cong M_{10}$.

Observe that $10, 16 \in \mathrm{cd}(G/L) \subseteq \mathrm{cd}(G)$ are the only even degrees in $\mathrm{cd}(G)$. Using the Atlas [4], we obtain $\tau(1)20 \in \mathrm{cd}(G \mid \tau)$, a third even degree. Thus, this case cannot occur.

For each possibility for G/L, we have shown that for all $\tau \in \mathrm{cd}(L)$, τ is linear and G-invariant, and $\mathrm{cd}(G \mid \tau) \subseteq \mathrm{cd}(G/L)$. We conclude that $L = Z(G)$ and $\mathrm{cd}(G) = \mathrm{cd}(G/L)$, as claimed. ☐

5.4 When q is Even

Now, we consider the possibility that the associated simple factor has even characteristic. We start with the case where $K/L \cong \mathrm{SL}_2(2^f)$ and K/L is properly contained in G/L.

Lemma 5.11. *Assume Hypothesis (*). If $q \geqslant 4$ is even and $K < G$, then $T = G$.*

Proof. By Theorem 1, $G/L \cong \mathrm{SL}_2(2^f) \rtimes Z_r$, where either r is an odd prime divisor of f or $r = f = 2$. We will assume $T < G$ and reach a contradiction in each case. Recall that by Lemma 5.6, we know that $K \nsubseteq T$ and so $T \cap K < K$.

We first suppose $r = f = 2$, so that $K/L \cong \mathrm{SL}_2(4) \cong A_5$ and $G/L \cong S_5$. We have

$$\mathrm{cd}(G/L) = \{1, 4, 5, 6\} \subseteq \mathrm{cd}(G),$$

hence 4 and 6 are the only even degrees of G. Thus if $|G : T|$ is even, then it must divide one of 4 or 6. But S_5 contains no subgroup of index 4 and since $T \neq K$ the index is not 2. If $|G : T| = 6$, then $\mathrm{cd}(T \mid \tau) = \{1\}$, and so τ extends to T and T/L is an abelian subgroup of S_5 of index 6, a contradiction.

Thus, $|G : T|$ is odd. It follows that $|G : T| = |K : T \cap K|$. The only proper subgroups of A_5 with odd index have index 5 or 15. Thus $|G : T|$ is 5 or 15, and so all degrees in $\mathrm{cd}(G \mid \tau)$ are divisible by 5. We conclude that no degree in $\mathrm{cd}(G \mid \tau)$ can be even, and so no degree in $\mathrm{cd}(T \mid \tau)$ is even, which implies that T/L has an abelian Sylow 2-subgroup, a contradiction.

We now consider the case where $G/L \cong \mathrm{SL}_2(2^f) \rtimes Z_r$ and r is an odd prime divisor of f. Since f is divisible by an odd prime, we see that $2^f \geqslant 8$, so we may apply Lemma 5.7 to determine that $(T \cap K)/L$ is not A_5. By Lemma 5.8, we know that $(T \cap K)/L$ is not $\mathrm{SL}_2(2^e)$ with e properly dividing f.

Suppose first that $r < f$, so that by Theorem 1, r divides neither $2^f - 1$ nor $2^f + 1$. In this case, we know that

$$\mathrm{cd}(G/L) = \{1, 2^f - 1, 2^f, 2^f + 1, (2^f - 1)r, (2^f + 1)r\}.$$

We also know that $|G : T| \geqslant |K : T \cap K| \geqslant 2^f + 1$ by Dickson's list. If $|K : T \cap K| > 2^f + 1$, then $|K : T \cap K|$, and hence $|G : T|$, is not in $\mathrm{cd}(G/L)$. On the other hand, $|K : T \cap K|$ will have a common prime divisor with either $2^f - 1$ or $2^f + 1$. If $c \in \mathrm{cd}(T \mid \tau)$, then we have either c, $2^f - 1$, $(2^f - 1)r$ or c, $2^f + 1$, $(2^f + 1)r$ are distinct degrees divisible by a common prime. We conclude that $|K : T \cap K| = 2^f + 1$.

We now assume $|K : T \cap K| = 2^f + 1$. It follows that $(T \cap K)/L$ is a Frobenius group. Let M/L be the Frobenius kernel of $(T \cap K)/L$. It follows that $|T \cap K : M| = 2^f - 1$ and $|M : L| = 2^f$. If τ does not extend to M, then there exists $b \in \mathrm{cd}(G \mid \tau)$ so that 2 divides b. We also have $2^f + 1$ dividing b, so $2^f + 1$, $(2^f + 1)r$, and b are all degrees in $\mathrm{cd}(G)$ with a common prime divisor. Hence τ extends to M. Notice that all the remaining Sylow subgroups of $(T \cap K)/L$ must be cyclic, so τ will extend to $T \cap K$. This implies that $(2^f - 1)\tau(1) \in \mathrm{cd}(K \cap T \mid \tau)$. Hence, there exists $a \in \mathrm{cd}(G \mid \tau)$ such that $(2^f - 1)(2^f + 1)$ divides a, and a, $2^f - 1$, $(2^f - 1)r$ are degrees in $\mathrm{cd}(G)$ with a common prime divisor. We conclude that $T = G$ in this case.

Finally, suppose $r = f$ is an odd prime. In this case, we know that

$$\mathrm{cd}(G/L) = \{1, 2^f - 1, 2^f, (2^f - 1)f, (2^f + 1)f\}.$$

If $(2^f - 1, |G : T|) > 1$, then $\mathrm{cd}(G \mid \tau) \subseteq \{2^f - 1, (2^f - 1)f\}$. It follows that $|G : T|$ divides $(2^f - 1)f$. We know that $|K : T \cap K| \geqslant 2^f + 1$ by Dickson's list, and $|K : T \cap K|$ divides $|G : T|$. We deduce that f divides $|K : T \cap K|$. The only case when f divides $|K : L| = |\mathrm{SL}_2(2^f)|$ is $f = 3$. We now have $(2^f - 1)f = 21$, but $\mathrm{SL}_2(8)$ has no proper subgroup of index dividing 21. We conclude that $|G : T|$ is relatively prime to $(2^f - 1)$.

Suppose that f divides $|G : T|$. This would imply that $\mathrm{cd}(G \mid \tau) \subseteq \{(2^f - 1)f, (2^f + 1)f\}$. As $|G : T|$ is relatively prime to $2^f - 1$, we conclude that $|G : T|$ divides $(2^f + 1)f$. Observe that either $|K : T \cap K|$ divides $2^f + 1$ or $f = 3$ and $(2^f + 1)f = 27$. Since $|K : T \cap K| \geqslant 2^f + 1$ and 27 does not divide $|\mathrm{SL}_2(8)|$, we conclude that $|K : T \cap K| = 2^f + 1$. By Dickson's list, this implies that $(T \cap K)/L$ is a Frobenius group. It follows that either $|G : T| = (2^f + 1)f$ or $f = 3$ and $|G : T| = |K : T \cap K| = 9$. Notice that if $|G : T| = (2^f + 1)f$, then $\mathrm{cd}(G \mid \tau) = \{(2^f + 1)f\}$ and $\mathrm{cd}(T \mid \tau) = \{1\}$. Hence, τ has only extensions to T, and so T/L is abelian, which is a contradiction since $(T \cap K)/L$ is a Frobenius group.

If $f = 3$ and $|K : T \cap K| = |G : T| = 9$, then we have $\mathrm{cd}(T \mid \tau) = \{3\}$. Let A/L be the Frobenius kernel of $(T \cap K)/L$, and note that A/L is the Sylow 2-subgroup of $(T \cap K)/L$. Since 2 divides no degree in $\mathrm{cd}(T \mid \tau)$, it follows that τ extends to A, and since the remaining Sylow subgroups are cyclic, τ will extend to T (see Corollaries 11.31 and 11.22 of [8]). This would imply by Gallagher's theorem that T/L is abelian, a contradiction. We conclude that f does not divide $|G : T|$.

We now have $|G : T| = |K : T \cap K|$ dividing $2^f(2^f + 1)$. This implies that $|T \cap K : L|$ is divisible by $2^f - 1$. The possibilities for $(T \cap K)/L$ are a cyclic group of order $2^f - 1$, a dihedral group of order $2(2^f - 1)$, and a Frobenius group of order $2^f(2^f - 1)$. Working with 2×2 matrices and the field automorphism, one can deduce that T/L is either a Frobenius group of order $(2^f - 1)f$ or $2(2^f - 1)f$, or an affine semi-linear group of order $2^f(2^f - 1)f$. (The affine semi-linear groups are defined in Section 2 of [13].)

If T/L is a Frobenius group, then all Sylow subgroups are cyclic, so τ will extend to T (see Corollaries 11.31 and 11.22 of [8]). It follows that $\mathrm{cd}(T \mid \tau) = \{a\tau(1) \mid a \in \mathrm{cd}(T/L)\}$ by Gallagher's theorem. If $|T/L| = (2^f - 1)f$, then $\mathrm{cd}(T \mid \tau) = \{\tau(1), f\tau(1)\}$ and as $|G : T| = 2^f(2^f + 1)$, we obtain

$$\mathrm{cd}(G \mid \tau) = \{2^f(2^f + 1)\tau(1), 2^f(2^f + 1)f\tau(1)\}.$$

If $|T/L| = 2(2^f - 1)f$, then $\mathrm{cd}(T \mid \tau) = \{\tau(1), 2f\tau(1)\}$ and as $|G : T| = 2^{f-1}(2^f + 1)$, we obtain

$$\mathrm{cd}(G \mid \tau) = \{2^{f-1}(2^f + 1)\tau(1), 2^f(2^f + 1)f\tau(1)\}.$$

In both cases, we obtain a contradiction by noticing that each of 2, f, and $2^f + 1$ divides three degrees in $\mathrm{cd}(G)$.

Suppose T/L is an affine semi-linear group of order $2^f(2^f - 1)f$. Let A/L be the normal subgroup of order 2^f, and let B/L be a subgroup of order $2^f - 1$. Observe that A/L is irreducible under the action of B/L, so A/L will be a chief factor of T. By Problem 6.12 of [8], either τ extends to A or τ is fully ramified with respect to A/L. Since f is odd, τ cannot be fully ramified, so τ extends to A. Since the other Sylow subgroups of T/L are cyclic, τ will extend to T (again, see Corollaries 11.31 and 11.22 of [8]). In Example 19.14 (c) of [7], it is shown that $\mathrm{cd}(T/L) = \{1, 2^f - 1, f\}$. We use Gallagher's theorem to see that $|\mathrm{cd}(T \mid \tau)| = 3$, contradicting Lemma 5.5, and we conclude that $T = G$. □

Note that since $\mathrm{PGL}_2(5) \cong S_5$, the following theorem includes one of the $q = 5$ cases not considered in Theorem 5.10.

Theorem 5.12. *Suppose that no prime divides three degrees in* $\mathrm{cd}(G)$ *and* G *has normal subgroups* $L < K$ *so that* $K/L \cong \mathrm{SL}_2(2^f)$. *If* $K < G$, *then* $L = Z(G)$ *and* $\mathrm{cd}(G) = \mathrm{cd}(G/L)$.

Proof. By Theorem 1, $G/L \cong \mathrm{SL}_2(2^f) \rtimes Z_r$, where either r is an odd prime divisor of f or $r = f = 2$. Let $\tau \in \mathrm{Irr}(L)$, so that by Lemma 5.11, τ is G-invariant.

We first suppose $r = f = 2$, so that $G/L \cong S_5$ and we have $\mathrm{cd}(G/L) = \{1, 4, 5, 6\} \subseteq \mathrm{cd}(G)$. The only possible even degrees in $\mathrm{cd}(G \mid \tau)$ are 4 and 6. Since τ is G-invariant, we may use the Atlas [4] to see that either $\mathrm{cd}(G \mid \tau) = \{\tau(1), 4\tau(1), 5\tau(1), 6\tau(1)\}$ or $\mathrm{cd}(G \mid \tau) = \{4\tau(1), 6\tau(1)\}$. Hence, we have $4\tau(1) \in \mathrm{cd}(G)$ in any case, and so $\tau(1) = 1$. This implies $\mathrm{cd}(G \mid \tau) \subseteq \{1, 4, 5, 6\}$, and so $\mathrm{cd}(G \mid \tau) \subseteq \mathrm{cd}(G/L)$.

We now consider the case where $G/L \cong \mathrm{SL}_2(2^f) \rtimes Z_r$ and r is an odd prime divisor of f. Since the Schur multiplier for $\mathrm{SL}_2(2^f)$ is trivial, we see that τ extends to K, and since K/L is simple, this extension is unique, so it is G-invariant. Since G/K is cyclic, we conclude that τ extends to G. In any case, 1, 2^f-1, and $(2^f-1)r$ are in $\mathrm{cd}(G/L)$, and so $\tau(1)$, $\tau(1)(2^f-1)$, and $\tau(1)(2^f-1)r$ are in $\mathrm{cd}(G)$. Thus, $\tau(1)=1$ and $\mathrm{cd}(G \mid \tau) = \mathrm{cd}(G/L)$.

We have shown that every irreducible character τ of L is G-invariant and linear, and that $\mathrm{cd}(G \mid \tau) \subseteq \mathrm{cd}(G/L)$. It follows that $L = Z(G)$ and $\mathrm{cd}(G) = \mathrm{cd}(G/L)$. ☐

The remaining case to consider is $G/L \cong \mathrm{SL}_2(2^f)$ where $f \geqslant 2$. Since $\mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$, this includes the remaining $q=5$ case not considered in Theorem 5.10. We first prove some preliminary lemmas.

Lemma 5.13. *Suppose no prime divides three degrees in* $\mathrm{cd}(G)$ *and* L *is normal in* G *so that* $G/L \cong \mathrm{SL}_2(2^f)$ *with* $f \geqslant 2$. *Consider* $\tau \in \mathrm{Irr}(L)$, *and let* T *be the stabilizer of* τ *in* G. *If* $T < G$, *then* $|G:T| \geqslant 2^f+1$. *Moreover, we have:*

1. *If* $|G:T| > 2^f+1$, *then* $\mathrm{cd}(G \mid \tau) = \{a\}$, *where* $a > 2^f+1$ *has common prime divisors with at least two of* 2^f-1, 2^f, 2^f+1.

2. *If* $|G:T| = 2^f+1$, *then* τ *is linear,* τ *extends to* T, *and* $\mathrm{cd}(G \mid \tau) = \{2^f+1, (2^f-1)(2^f+1)\}$.

Proof. Since $|G:T| > 1$, it follows from Lemma 5.5 that $|\mathrm{cd}(G \mid \tau)| \leqslant 2$. By Dickson's list, we know that $|G:T| \geqslant 2^f+1$.

Suppose first that $|G:T| > 2^f+1$. If $a \in \mathrm{cd}(G \mid \tau)$, then $a \geqslant |G:T| > 2^f+1$. If p is any prime divisor of $|G:T|$, then a is divisible by p and p divides one of 2^f-1, 2^f, 2^f+1. Suppose we have $a, b \in \mathrm{cd}(G \mid \tau)$ with $a < b$. We know that p divides both a and b. Thus, p will divide three degrees in $\mathrm{cd}(G)$, a contradiction. Hence, we conclude that $\mathrm{cd}(G \mid \tau) = \{a\}$. Since $|G:T|$ divides $(2^f-1)2^f(2^f+1)$ and is greater than 2^f+1, it must have common prime divisors with at least two of the numbers 2^f-1, 2^f, and 2^f+1, and hence a does also.

Suppose now $|G:T| = 2^f+1$. Using Clifford theory, we obtain $1 \in \mathrm{cd}(T \mid \tau)$. We conclude that τ must extend to T. We know that T/L is a Frobenius group and $\mathrm{cd}(T/L) = \{1, 2^f-1\}$. We now apply Gallagher's theorem to see that $\mathrm{cd}(T \mid \tau) = \{1, 2^f-1\}$. It follows that $\mathrm{cd}(G \mid \tau) = \{2^f+1, (2^f-1)(2^f+1)\}$. ☐

The next lemma is the key to our argument.

Lemma 5.14. *Suppose no prime divides three degrees in* $\mathrm{cd}(G)$ *and* L *is normal in* G *so that* $G/L \cong \mathrm{SL}_2(2^f)$ *with* $f \geqslant 2$. *If* $\mu, \nu \in \mathrm{Irr}(L)$ *are both not* G*-invariant, then*

$$\{2^f+1\} \cup \mathrm{cd}(G \mid \mu) = \{2^f+1\} \cup \mathrm{cd}(G \mid \nu).$$

Proof. We apply Lemma 5.13 to see that there exist a_1, a_2 so that

$$\{2^f+1\} \cup \mathrm{cd}(G \mid \mu) = \{2^f+1, a_1\} \text{ and } \{2^f+1\} \cup \mathrm{cd}(G \mid \nu) = \{2^f+1, a_2\},$$

with $a_i > 2^f+1$ for $i = 1, 2$. It suffices to show that $a_1 = a_2$. Thus, we assume that $a_1 \neq a_2$. Let T be the stabilizer of μ in G and S the stabilizer of ν in G. We know that $|G:T|$ and $|G:S|$ divide $|G:L| = (2^f-1)2^f(2^f+1)$. Hence, if $|G:T|$ and $|G:S|$ have a common prime divisor p, then p will divide one of the degrees 2^f-1, 2^f, 2^f+1 of G, as well as the degrees a_1 and a_2, a contradiction. Thus, $|G:T|$ and $|G:S|$ must be relatively prime. In particular, one of $|G:T|$ or $|G:S|$ must be odd.

Without loss of generality, let $|G:T|$ be odd. If $|G:T| = (2^f-1)(2^f+1)$, then as $|G:S|$ is relatively prime to $|G:T|$, we conclude that $|G:S|$ divides 2^f, which contradicts the fact that $\mathrm{SL}_2(2^f)$ has no proper subgroups of 2-power index. Thus, $|G:T| < (2^f-1)(2^f+1)$. It follows that

T/L is not abelian. Notice that T/L contains a full Sylow 2-subgroup of G/L. From Dickson's list of subgroups, this implies that either T/L is isomorphic to A_5 or T/L is a Frobenius group whose Frobenius kernel is the Sylow 2-subgroup of G/L contained in T/L (including possibly A_4). Since $T < G$, we know that if $T/L \cong A_5$, then $f \geqslant 3$, so $2^f \geqslant 8$. We may use Lemma 5.7 to see that T/L is not isomorphic to A_5.

Suppose that T/L is a Frobenius group. Let P/L be the Sylow 2-subgroup of T/L. We know that $\mathrm{cd}(T/L) = \{1, |T : P|\}$. If μ extends to P, then since the other Sylow subgroups of T/K are cyclic, μ extends to T. By applying Gallagher's theorem, we have that $|T : P|$ divides some degree in $\mathrm{cd}(T \mid \mu)$. It follows that $|G' : P| = (2^f - 1)(2^f + 1)$ divides some degree in $\mathrm{cd}(G \mid \mu)$, and hence $(2^f - 1)(2^f + 1)$ divides a_1. This implies that a_2, and hence $|G : S|$, must be relatively prime to $(2^f - 1)(2^f + 1)$. We conclude that $|G : S|$ is a power of 2, a contradiction.

If μ does not extend to P, then 2 divides the degrees in $\mathrm{cd}(P \mid \mu)$, and hence 2 divides the degrees in $\mathrm{cd}(G \mid \mu)$. In particular 2 divides a_1. If 2 divides a_2, then a_1, a_2, and 2^f are three distinct even degrees of G, and so we must have that a_2 is odd. Hence, $|G : S|$ is odd, and so $|G : S|$ divides $(2^f+1)(2^f-1)$. Since $|G : T|$ and $|G : S|$ are coprime and both divide $(2^f+1)(2^f-1)$, we conclude that $|G : T| \cdot |G : S|$ divides $(2^f+1)(2^f-1)$. On the other hand, the fact that $|G : T| \geqslant 2^f+1$ and $|G : S| \geqslant 2^f + 1$ implies that

$$|G : T| \cdot |G : S| \geqslant (2^f + 1)^2 > (2^f + 1)(2^f - 1),$$

a contradiction. Hence $a_1 = a_2$ and the result follows. □

Finally, we prove the desired result when $G/L \cong \mathrm{SL}_2(2^f)$. We should note that if G is the semi-direct product of $\mathrm{SL}_2(2^f)$ acting on its natural module, then

$$\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, 2^f + 1, (2^f - 1)(2^f + 1)\}.$$

We do not know of any other examples where this occurs. In particular, we do not know of any examples where a in the following theorem has a value other than $(2^f - 1)(2^f + 1)$.

Theorem 5.15. *If no prime divides three degrees in* $\mathrm{cd}(G)$ *and* L *is a normal subgroup of* G *so that* $G/L \cong \mathrm{SL}_2(2^f)$ *with* $f \geqslant 2$, *then* $\mathrm{cd}(G) \subseteq \{1, 2^f - 1, 2^f, 2^f + 1, a\}$, *where* $a > 2^f + 1$.

Proof. Recall that $\mathrm{cd}(G/L) = \{1, 2^f-1, 2^f, 2^f+1\}$. Suppose the character $\mu \in \mathrm{Irr}(L)$ is G-invariant. If μ extends to G, then

$$\{\mu(1), \mu(1)(2^f - 1), \mu(1)2^f, \mu(1)(2^f + 1)\} \subseteq \mathrm{cd}(G \mid \mu).$$

This implies $\mu(1) = 1$ and $\mathrm{cd}(G \mid \mu) = \{1, 2^f - 1, 2^f, 2^f + 1\}$. If μ does not extend to G, then $f = 2$ and $G/L \cong \mathrm{SL}_2(4)$ since the Schur multiplier of $\mathrm{SL}_2(2^f)$ is trivial for $f > 2$. By the Atlas [4], we obtain

$$\{2\mu(1), 4\mu(1), 6\mu(1)\} \subseteq \mathrm{cd}(G \mid \mu),$$

a contradiction. Thus, we have $\mathrm{cd}(G \mid \mu) = \{1, 2^f - 1, 2^f, 2^f + 1\}$ for every invariant character $\mu \in \mathrm{Irr}(L)$.

Suppose now that $\mu \in \mathrm{Irr}(L)$ is not G-invariant. We apply Lemma 5.13 to see that

$$\{2^f + 1\} \cup \mathrm{cd}(G \mid \mu) = \{2^f + 1, a\},$$

where $a > 2^f+1$ and a has common prime divisors with at least two of the numbers $2^f-1, 2^f, 2^f+1$. If ν is any other character in $\mathrm{Irr}(L)$ that is not G-invariant, then we apply Corollary 5.14 to see that $\mathrm{cd}(G \mid \nu) \subseteq \{2^f + 1, a\}$. It follows that $\mathrm{cd}(G) \subseteq \{1, 2^f - 1, 2^f, 2^f + 1, a\}$. □

5.5 Conclusions

We have shown that if G is a nonsolvable group where no prime divides three degrees of G, then G has normal subgroups $L < K$ such that $K/L \cong \mathrm{PSL}_2(q)$, for some prime power $q \geqslant 4$, and $G/L \leqslant \mathrm{Aut}\, K/L \cong \mathrm{Aut}\, \mathrm{PSL}_2(q)$. By Theorem 5.10 and Theorem 5.12, we have that if $q > 5$ is odd or if $q \geqslant 4$ is even with $K < G$, then $\mathrm{cd}(G) = \mathrm{cd}(G/L)$. Thus Table 1 shows that $|\mathrm{cd}(G)| \leqslant 6$ in these cases. If $q \geqslant 4$ is even and $K = G$, so that $G/L \cong \mathrm{PSL}_2(q)$, then Theorem 5.15 shows that $\mathrm{cd}(G) \subseteq \mathrm{cd}(G/L) \cup \{a\}$ for some $a > q + 1$. Hence $|\mathrm{cd}(G)| \leqslant 5$ in this case. Therefore, if G is any nonsolvable group where no prime divides three degrees, then $|\mathrm{cd}(G)| \leqslant 6$. As Benjamin [1] had shown this previously for solvable groups, Theorem 2 follows.

As noted previously, if $\Gamma(G)$ contains no triangles, then no prime can divide three degrees of G, and so Theorem 3 holds. However, there exist solvable groups for which no prime divides three degrees and yet $\Gamma(G)$ contains a triangle. Theorem 1 and the list of degree sets in Table 1 show that if no prime divides three degrees of a nonsolvable group G, then $\Gamma(G/L)$ contains no triangles. Hence if $q > 5$ is odd or if $q \geqslant 4$ is even with $K < G$, it follows that $\Gamma(G)$ contains no triangles. If $q \geqslant 4$ is even and $K = G$, then $\mathrm{cd}(G) \subseteq \{1, 2^f - 1, 2^f, 2^f + 1, a\}$ for some $a > 2^f + 1$. Since $2^f - 1$, 2^f, and $2^f + 1$ are pairwise coprime, $\Gamma(G)$ also contains no triangles in this case, hence Theorem 4 follows.

References

[1] D. Benjamin, Coprimeness among irreducible character degrees of finite solvable groups, *Proc. AMS* **125** (1997), 2831-2837.

[2] M. Bianchi, D. Chillag, M. L. Lewis, and E. Pacifici, Character degree graphs that are complete graphs, *Proc. AMS* **135** (2007), 671-676.

[3] R. W. Carter, "Finite Groups of Lie Type: Conjugacy Classes and Complex Characters," Wiley, New York, 1985.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Oxford University Press, London, 1984.

[5] L. Dornhoff, "Group Representation Theory, Part A: Ordinary Representation Theory," Marcel Dekker, New York, 1971.

[6] B. Huppert, "Endliche Gruppen I," Springer-Verlag, Berlin, 1983.

[7] B. Huppert, "Character Theory of Finite Groups," Walter de Gruyter, Berlin, 1998.

[8] I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, San Diego, 1976.

[9] M. L. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mtn. J. Math.* **38** (2008), 175-211.

[10] M. L. Lewis, A. Moretó, and T. Wolf, Non-divisibility among character degrees, *J. Group Theory* **8** (2008), 561-588.

[11] T. Li, Y. Liu, and X. Song, Finite nonsolvable groups whose character graphs have no triangles, *J. Algebra* **323** (8) (2010), 2290–2300.

[12] O. Manz, R. Staszewski, and W. Willems, On the number of components of a graph related to character degrees, *Proc. AMS* **103** (1988), 31–37.

[13] O. Manz and T. R. Wolf, "Representation Theory of Solvable Groups" Cambridge University Press, Cambridge, 1993.

[14] J. K. McVey, Prime divisibility among degrees of solvable groups, *Comm. Alg.* **32** (2004), 3391-3402.

[15] P. Schmid, Extending the Steinberg representation, *J. Algebra* **150** (1992), 254–256.

[16] W. A. Simpson and J. S. Frame, The character tables for $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, *Can. J. Math.* **25** (1973), 486–494.

[17] R. Steinberg, The representations of $\mathrm{GL}(3,q)$, $\mathrm{GL}(4,q)$, $\mathrm{PGL}(3,q)$, and $\mathrm{PGL}(4,q)$, *Can. J. Math.* **3** (1951), 225–235.

[18] M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* **75** (1962), 105–145.

[19] Y.-T. Wu and P. Zhang, Finite solvable groups whose character graphs are trees, *J. Algebra* **308** (2) (2007), 536–544.