

ALGEBRA QUALIFYING EXAM PROBLEMS  
GROUP THEORY

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# GROUP THEORY

## General Group Theory

1. Prove or give a counter-example:
  - (a) If  $H_1$  and  $H_2$  are groups and  $G = H_1 \times H_2$ , then any subgroup of  $G$  is of the form  $K_1 \times K_2$ , where  $K_i$  is a subgroup of  $H_i$  for  $i = 1, 2$ .
  - (b) If  $H \trianglelefteq N$  and  $N \trianglelefteq G$  then  $H \trianglelefteq G$ .
  - (c) If  $G_1 \cong H_1$  and  $G_2 \cong H_2$ , then  $G_1 \times G_2 \cong H_1 \times H_2$ .
  - (d) If  $N_1 \trianglelefteq G_1$  and  $N_2 \trianglelefteq G_2$  with  $N_1 \cong N_2$  and  $G_1/N_1 \cong G_2/N_2$ , then  $G_1 \cong G_2$ .
  - (e) If  $N_1 \trianglelefteq G_1$  and  $N_2 \trianglelefteq G_2$  with  $G_1 \cong G_2$  and  $N_1 \cong N_2$ , then  $G_1/N_1 \cong G_2/N_2$ .
  - (f) If  $N_1 \trianglelefteq G_1$  and  $N_2 \trianglelefteq G_2$  with  $G_1 \cong G_2$  and  $G_1/N_1 \cong G_2/N_2$ , then  $N_1 \cong N_2$ .
2. Let  $G$  be a group and let  $N$  be a normal subgroup of index  $n$ . Show that  $g^n \in N$  for all  $g \in G$ .
3. Let  $G$  be a group. A subgroup  $H$  of  $G$  is called a *characteristic* subgroup of  $G$  if  $\varphi(H) = H$  for every automorphism  $\varphi$  of  $G$ . Show that if  $H$  is a characteristic subgroup of  $N$  and  $N$  is a normal subgroup of  $G$ , then  $H$  is a normal subgroup of  $G$ .
4. Show that if  $H$  is a characteristic subgroup of  $N$  and  $N$  is a characteristic subgroup of  $G$ , then  $H$  is a characteristic subgroup of  $G$ .
5. Let  $G$  be a finite group,  $H$  a subgroup of  $G$  and  $N$  a normal subgroup of  $G$ . Show that if the order of  $H$  is relatively prime to the index of  $N$  in  $G$ , then  $H \subseteq N$ .
6. Let  $G$  be a group and let  $Z(G)$  be its center. Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.
7. Let  $G$  be a group and let  $Z(G)$  be the center of  $G$ . Prove or disprove the following.
  - (a) If  $G/Z(G)$  is cyclic, then  $G$  is abelian.
  - (b) If  $G/Z(G)$  is abelian, then  $G$  is abelian.
  - (c) If  $G$  is of order  $p^2$ , where  $p$  is a prime, then  $G$  is abelian.
8. Show that if  $G$  is a nonabelian finite group, then  $|Z(G)| \leq \frac{1}{4}|G|$ .
9. Let  $G$  be a finite group and let  $M$  be a maximal subgroup of  $G$ . Show that if  $M$  is a normal subgroup of  $G$ , then  $|G : M|$  is prime.
10. Show that if  $\mathcal{K}$  and  $\mathcal{L}$  are conjugacy classes of groups  $G$  and  $H$ , respectively, then  $\mathcal{K} \times \mathcal{L}$  is a conjugacy class of  $G \times H$ .
11.
  - (a) State a formula relating orders of centralizers and cardinalities of conjugacy classes in a finite group  $G$ .
  - (b) Let  $G$  be a finite group with a proper normal subgroup  $N$  that is not contained in the center of  $G$ . Prove that  $G$  has a proper subgroup  $H$  with  $|H| > |G|^{1/2}$ . [Hint: (a) applied to a noncentral element of  $G$  inside  $N$  is useful.]
12. Let  $H$  be a subgroup of  $G$  of index 2 and let  $g$  be an element of  $H$ . Show that if  $C_G(g) \subseteq H$  then the conjugacy class of  $g$  in  $G$  splits into 2 conjugacy classes in  $H$ , and if  $C_G(g) \not\subseteq H$ , then the class of  $g$  in  $G$  remains the class of  $g$  in  $H$ .

13. Let  $G$  be a finite group,  $H$  a subgroup of  $G$  of index 2, and  $x \in H$ . Denote by  $c\ell_G(x)$  the conjugacy class of  $x$  in  $G$  and by  $c\ell_H(x)$  the conjugacy class of  $x$  in  $H$ .
- Show that if  $C_G(x) \leq H$ , then  $|c\ell_H(x)| = \frac{1}{2}|c\ell_G(x)|$ .
  - Show that if  $C_G(x)$  is not contained in  $H$ , then  $|c\ell_H(x)| = |c\ell_G(x)|$ .
- [Hint: Consider centralizer orders.]
14. Let  $N$  be a normal subgroup of  $G$  and let  $\mathcal{C}$  be a conjugacy class of  $G$  that is contained in  $N$ . Prove that if  $|G : N| = p$  is prime, then either  $\mathcal{C}$  is a conjugacy class of  $N$  or  $\mathcal{C}$  is a union of  $p$  distinct conjugacy classes of  $N$ .
15. Let  $G$  be a group,  $g \in G$  an element of order greater than 2 (possibly infinite) such that the conjugacy class of  $g$  has an odd number of elements. Prove that  $g$  is not conjugate to  $g^{-1}$ .
16. Let  $H$  be a subgroup of the group  $G$ . Show that the following are equivalent:
- $x^{-1}y^{-1}xy \in H$  for all  $x, y \in G$
  - $H \trianglelefteq G$  and  $G/H$  is abelian.
17. Let  $H$  and  $K$  be subgroups of a group  $G$ , with  $K \trianglelefteq G$  and  $K \leq H$ . Show that  $H/K$  is contained in the center of  $G/K$  if and only if  $[H, G] \leq K$  (where  $[H, G] = \langle h^{-1}g^{-1}hg \mid h \in H, g \in G \rangle$ ).
18. Let  $G$  be any group for which  $G'/G''$  and  $G''/G'''$  are cyclic. Prove that  $G'' = G'''$ .
19. Let  $\text{GL}_n(\mathbb{C})$  be the group of invertible  $n \times n$  matrices with complex entries. Give a complete list of conjugacy class representatives for  $\text{GL}_2(\mathbb{C})$  and for  $\text{GL}_3(\mathbb{C})$ .
20. Let  $H$  be a subgroup of the group  $G$  and let  $T$  be a set of representatives for the distinct right cosets of  $H$  in  $G$ . In particular, if  $t \in T$  and  $g \in G$  then  $tg$  belongs to a unique coset of the form  $Ht'$  for some  $t' \in T$ . Write  $t' = t \cdot g$ . Prove that if  $S \subseteq G$  generates  $G$ , then the set  $\{ts(t \cdot s)^{-1} \mid t \in T, s \in S\}$  generates  $H$ .  
Suggestion: If  $K$  denotes the subgroup generated by this set, prove the stronger assertion that  $KT = G$ . Start by showing that  $KT$  is stable under right multiplication by elements of  $G$ .
21. Let  $G$  be a group,  $H$  a subgroup of finite index  $n$ ,  $G/H$  the set of left cosets of  $H$  in  $G$ , and  $S(G/H)$  the group of permutations of  $G/H$  (with composition from right to left). Define  $f : G \rightarrow S(G/H)$  by  $f(g)(xH) = (gx)H$  for  $g, x \in G$ .
- Show that  $f$  is a group homomorphism.
  - Show that if  $H$  is a normal subgroup of  $G$ , then  $H$  is the kernel of  $f$ .
22. Let  $G$  be an abelian group. Let  $K = \{a \in G : a^2 = 1\}$  and let  $H = \{x^2 : x \in G\}$ . Show that  $G/K \cong H$ .
23. Let  $N \trianglelefteq G$  such that every subgroup of  $N$  is normal in  $G$  and  $C_G(N) \subseteq N$ . Prove that  $G/N$  is abelian.
24. Let  $H$  be a subgroup of  $G$  having a normal complement (i.e., a normal subgroup  $N$  of  $G$  satisfying  $HN = G$  and  $H \cap N = \langle 1 \rangle$ ). Prove that if two elements of  $H$  are conjugate in  $G$ , then they are conjugate in  $H$ .

25. Let  $H$  be a subgroup of the group  $G$  with the property that whenever two elements of  $G$  are conjugate, then the conjugating element can be chosen within  $H$ . Prove that the commutator subgroup  $G'$  of  $G$  is contained in  $H$ .
26. Let  $a \in G$  be fixed, where  $G$  is a group. Prove that  $a$  commutes with each of its conjugates in  $G$  if and only if  $a$  belongs to an abelian normal subgroup of  $G$ .
27. Show that if  $H$  and  $K$  are subgroups of a finite group  $G$  satisfying  $(|G : H|, |G : K|) = 1$ , then  $G = HK$ .
28. Let  $G = A \times B$  be a direct product of the subgroups  $A$  and  $B$ . Suppose  $H$  is a subgroup of  $G$  that satisfies  $HA = G = HB$  and  $H \cap A = \langle 1 \rangle = H \cap B$ . Prove that  $A$  is isomorphic to  $B$ .
29. Let  $N_1, N_2$ , and  $N_3$  be normal subgroups of a group  $G$  and assume that for  $i \neq j$ ,  $N_i \cap N_j = \langle 1 \rangle$  and  $N_i N_j = G$ . Show that  $G$  is isomorphic to  $N_1 \times N_2$  and  $G$  is abelian.
30. Show that if the size of each conjugacy class of a group  $G$  is at most 2, then  $G' \leq Z(G)$ .
31. Let  $N$  be a normal subgroup of  $G$ . Show that if  $N \cap G' = \langle 1 \rangle$ , then  $N$  is contained in the center of  $G$ .
32. Let  $G$  be a finite group.
  - (a) Show that every proper subgroup of  $G$  is contained in a maximal subgroup.
  - (b) Show that the intersection of all maximal subgroups of  $G$  is a normal subgroup.
33. Let  $G$  be a group. Show that  $G$  has a composition series if and only if  $G$  satisfies the following two conditions:
  - (i) If  $G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$  is any subnormal series, then there is an  $n$  such that  $H_n = H_{n+1} = \cdots$ .
  - (ii) If  $H$  is any subgroup of  $G$  in a subnormal series and  $K_1 \leq K_2 \leq K_3 \leq \cdots$  is an ascending chain of normal subgroups of  $H$ , then there is an  $m$  such that  $K_m = K_{m+1} = \cdots$ .

## Cyclic Groups

34. Let  $\varphi$  be the Euler  $\varphi$ -function — that is,  $\varphi(n)$  is the number of positive integers less than the integer  $n$  and relatively prime to  $n$ . Let  $G$  be a finite group of order  $n$  with at most  $d$  elements  $x$  satisfying  $x^d = 1$  for each divisor  $d$  of  $n$ .
  - (a) Show that in a *cyclic* group of order  $n$ , the number of elements of order  $d$  is  $\varphi(d)$  for each divisor  $d$  of  $n$ . Deduce that  $\sum_{d|n} \varphi(d) = n$ .
  - (b) Let  $\psi(d)$  be the number of elements of  $G$  of order  $d$ . Show that for any  $d$ , either  $\psi(d) = 0$  or  $\psi(d) = \varphi(d)$ .
  - (c) Show that  $G$  is cyclic.
  - (d) Show that any finite subgroup of the multiplicative group of a field must be cyclic.
35. Show that if  $G$  is a cyclic group then every subgroup of  $G$  is cyclic.
36. Show that if  $G$  is a finite cyclic group, then  $G$  has exactly one subgroup of order  $m$  for each positive integer  $m$  dividing  $|G|$ .

37. Show that if  $H$  is a cyclic normal subgroup of a finite group  $G$ , then every subgroup of  $H$  is a normal subgroup of  $G$ .
38. Let  $G$  be a cyclic group of order 12 with generator  $a$ . Find  $b$  in  $G$  such that  $G/\langle b \rangle$  is isomorphic to  $\langle a^{10} \rangle$ . (Here  $\langle x \rangle$  denotes the subgroup of  $G$  generated by  $\{x\}$ , for  $x \in G$ .)

## Homomorphisms

39. State and prove the three “isomorphism theorems” (for groups).
40. Let  $G$  be a group and let  $K$  be a subgroup of  $G$ . Give necessary and sufficient conditions for  $K$  to be the kernel of a homomorphism from  $G$  to  $G$ . Prove your answer. (*N.B.*: The homomorphism must be from  $G$  to  $G$ .)
41. Let  $G$  be a group with a normal subgroup  $N$  of order 5, such that  $G/N \cong S_3$ . Show that  $|G| = 30$ ,  $G$  has a normal subgroup of order 15, and  $G$  has 3 subgroups of order 10 that are not normal.
42. Let  $G$  be a group with a normal subgroup  $N$  of order 7, such that  $G/N \cong D_{10}$ , the dihedral group of order 10. Show that  $|G| = 70$ ,  $G$  has a normal subgroup of order 35, and  $G$  has 5 subgroups of order 14 that are not normal.
43. Let  $f : G \rightarrow H$  be a homomorphism of groups with kernel  $K$  and image  $I$ .
- (a) Show that if  $N$  is a subgroup of  $G$  then  $f^{-1}(f(N)) = KN$ .
- (b) Show that if  $L$  is a subgroup of  $H$  then  $f(f^{-1}(L)) = I \cap L$ .
44. Let  $G$  and  $H$  be finite groups with  $(|G|, |H|) = 1$ . Show that if  $\varphi : G \rightarrow H$  is a homomorphism, then  $\varphi(g) = 1_H$  for all  $g$  in  $G$  (where  $1_H$  is the identity element of  $H$ ).
45. Let  $G = \text{GL}_n(\mathbb{R})$  be the (multiplicative) group of nonsingular  $n \times n$  matrices with real entries and let  $S = \text{SL}_n(\mathbb{R})$  be the subgroup of  $G$  consisting of matrices of determinant 1. Show that  $S \trianglelefteq G$  and  $G/S \cong \mathbb{R}^*$ , the multiplicative group of real numbers.
46. (a) Suppose  $H$  and  $K$  are normal subgroups of a group  $G$ . Show that there exists a one-to-one homomorphism

$$\varphi : G/H \cap K \rightarrow G/H \times G/K.$$

- (b) Use part (a) to show that if  $(m, n) = 1$  then  $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ .
47. Prove that the commutator subgroup of  $\text{SL}_2(\mathbb{Z})$  is *proper* in  $\text{SL}_2(\mathbb{Z})$ . (Hint: Any homomorphism of rings  $R \rightarrow S$  induces a homomorphism of groups  $\text{SL}_2(R) \rightarrow \text{SL}_2(S)$ .)
48. Let  $H$  and  $K$  be subgroups of a finite group  $G$  and assume  $H$  is isomorphic to  $K$ . Prove that there exists a group  $\tilde{G}$  containing  $G$  as a subgroup, such that  $H$  and  $K$  are conjugate in  $\tilde{G}$ .

## Automorphism Groups

49. Let  $\text{Inn}(G)$  be the group of inner automorphisms of the group  $G$  and let  $\text{Aut}(G)$  be the full automorphism group.
- (a) Show that  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$ .
- (b) Show that if  $Z(G)$  is the center of  $G$ , then  $\text{Inn}(G) \cong G/Z(G)$ .

50. Show that if  $H$  is a subgroup of  $G$ , then  $C_G(H) \trianglelefteq N_G(H)$  and  $N_G(H)/C_G(H)$  is isomorphic to a subgroup of  $\text{Aut}(H)$ .
51. Let  $G$  be a simple group of order greater than 2 and let  $\text{Aut}(G)$  be its automorphism group. Show that the center of  $\text{Aut}(G)$  is trivial if and only if  $G$  is non-abelian.
52. Let  $G$  be a finite group with a normal subgroup  $N \cong S_3$ . Show that there is a subgroup  $H$  of  $G$  such that  $G = N \times H$ .
53. A group  $N$  is said to be *complete* if the center of  $N$  is trivial and every automorphism of  $N$  is inner. Show that if  $G$  is a group,  $N \trianglelefteq G$ , and  $N$  is complete, then  $G = N \times C_G(N)$ .

## Abelian Groups

54. Let  $A$  be an abelian group with the following property:  
 (\*) If  $B \leq A$  then there is a  $C \leq A$  with  $A = B \oplus C$ .  
 Show the following.
- (a) Each subgroup of  $A$  satisfies (\*).  
 (b) Each element of  $A$  has finite order.  
 (c) If  $p$  is a prime, then  $A$  has no element of order  $p^2$ .
55. Let  $A$  be an abelian  $p$ -group of exponent  $p^m$ . Show that if  $B$  is a subgroup of  $A$  of order  $p^m$  and both  $B$  and  $A/B$  are cyclic, then there is a subgroup  $C$  of  $A$  such that  $A = B + C$  and  $B \cap C = \{0\}$ .
56. (a) List all abelian groups of order 360 (up to isomorphism).  
 (b) Find the invariant factors and elementary divisors of the group

$$G = \mathbb{Z}_{25} \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{48} \oplus \mathbb{Z}_{300}.$$

57. Consider the property (\*) of abelian groups  $G$ :

(\*) If  $H$  is any subgroup of  $G$  then there exists a subgroup  $F$  of  $G$  such that  $G/H \cong F$ .

Show that if  $G$  is a finitely generated abelian group then  $G$  has property (\*) if and only if  $G$  is finite.

58. Let  $n$  be a positive integer and let  $A = \mathbb{Z}^n$ . Prove that if  $B$  is any subgroup of  $A$  that is generated by fewer than  $n$  elements, then the index  $[A : B]$  is infinite.
59. Show that if  $A$ ,  $B$ , and  $C$  are abelian groups, then

$$\text{Hom}(A, B \oplus C) \cong \text{Hom}(A, B) \oplus \text{Hom}(A, C).$$

60. Show that if  $A$ ,  $B$ , and  $C$  are abelian groups, then

$$\text{Hom}(A \oplus B, C) \cong \text{Hom}(A, C) \oplus \text{Hom}(B, C).$$

61. Let  $A, B, A_\alpha$  ( $\alpha \in I$ ) and  $B_\beta$  ( $\beta \in J$ ) be abelian groups. Prove the following:

$$\text{Hom}\left(\bigoplus_{\alpha \in I} A_\alpha, B\right) \cong \prod_{\alpha \in I} \text{Hom}(A_\alpha, B)$$

$$\text{Hom}\left(A, \prod_{\beta \in J} B_\beta\right) \cong \prod_{\beta \in J} \text{Hom}(A, B_\beta).$$

62. Let:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of Abelian groups and homomorphisms in which both rows are exact. If  $\alpha, \beta, \delta,$  and  $\epsilon$  are isomorphisms, prove that  $\gamma$  is an isomorphism also.

63. Let  $A, U, V, W, X,$  and  $Y$  be abelian groups.

If  $\alpha \in \text{Hom}(X, Y)$  define  $\alpha_* : \text{Hom}(A, X) \rightarrow \text{Hom}(A, Y)$  by  $\alpha_*(f) = \alpha \circ f$ . If

$$0 \rightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \rightarrow 0$$

is exact, to what extent is

$$0 \rightarrow \text{Hom}(A, U) \xrightarrow{\alpha_*} \text{Hom}(A, V) \xrightarrow{\beta_*} \text{Hom}(A, W) \rightarrow 0$$

exact? Prove your assertions.

64. Same as the previous problem, except use  $\text{Hom}(-, A)$  instead, making the obvious modifications.

## Symmetric Groups

65. (a) Find the centralizer in  $S_7$  of  $(1\ 2\ 3)(4\ 5\ 6\ 7)$ .  
 (b) How many elements of order 12 are there in  $S_7$ ?
66. (a) Give an example of two nonconjugate elements of  $S_7$  that have the same order.  
 (b) If  $g \in S_7$  has maximal order, what is  $o(g)$ ?  
 (c) Does the element  $g$  that you found in part (b) lie in  $A_7$ ?  
 (d) Is the set  $\{h \in S_7 \mid o(h) = o(g)\}$  a single conjugacy class in  $S_7$ , where  $g$  is the element found in part (b)?
67. (a) Give a representative for each conjugacy class of elements of order 6 in  $S_6$ .  
 (b) Find the order of the centralizer in  $S_6$  of each element from part (a).
68. How many elements of order 6 are there in  $S_6$ ? How many in  $A_6$ ?
69. (a) Write  $\sigma = (4\ 5\ 6)(2\ 3)(1\ 2)(6\ 7\ 8)$  as a product of disjoint cycles and find the order of  $\sigma$ .  
 (b) Let  $n > 1$  be an odd integer. Show that  $S_n$  has an element of order  $2(n-2)$ .

70. Let  $\sigma = (1\ 2\ 3)(4\ 5\ 6) \in S_6$ .
- Determine the size of the conjugacy class of  $\sigma$  and the order of the centralizer of  $\sigma$  in  $S_6$ .
  - Determine if  $C_{S_6}(\sigma)$  is abelian or non-abelian. Prove your answer.
71. Let  $G$  be a subgroup of the symmetric group  $S_n$ . Show that if  $G$  contains an odd permutation then  $G \cap A_n$  is of index 2 in  $G$ .
72. Show that if  $G$  is a non-abelian simple subgroup of  $S_n$ , then  $G$  is contained in  $A_n$ .
73. Show that if  $G$  is a subgroup of  $S_n$  of index 2, then  $G = A_n$ .
74. For  $i = 1, \dots, n-1$ , let  $x_i$  be the transposition  $(i\ i+1)$  in the symmetric group  $S_n$ . Show that  $S_n = \langle x_1, \dots, x_{n-1} \rangle$ .
75. Let  $H$  be a subgroup of  $S_n$ . Show that if  $H$  is a transitive subgroup of  $S_n$  and  $H$  is generated by some set of transpositions, then  $H = S_n$ .
76. Prove that the symmetric group  $S_n$  is a maximal subgroup of  $S_{n+1}$ .  
[Hint: Show that if  $g \in S_{n+1} - S_n$ , then  $S_{n+1} = S_n \cup S_n g S_n$ .]
77. (a) If  $n = k + \ell$  with  $k \neq \ell$ , then  $S_k \times S_\ell$  is a maximal subgroup of  $S_n$  in the natural embedding.  
(b) If  $n = 2k$ , then  $S_k \times S_k$  is not a maximal subgroup of  $S_n$  in the natural embedding.
78. (a) Prove that if  $A$  is a transitive abelian subgroup of the symmetric group  $S_n$ , then  $|A| = n$ .  
(b) Give an example of  $n$ ,  $A_1$ ,  $A_2$ , where  $A_1$  and  $A_2$  are transitive abelian subgroups of  $S_n$ , but  $A_1$  is not isomorphic to  $A_2$ .
79. Let  $A$  be an abelian, transitive subgroup of  $S_n$ . Show that for all  $\alpha \in \{1, \dots, n\}$ , the stabilizer  $A_\alpha$  of  $\alpha$  in  $A$  is trivial.
80. Let  $H$  be a subgroup of index  $n$  in a group  $G$ . Let  $S_n$  be the symmetric group on  $n$  letters and let  $S_{n-1} \subseteq S_n$  be the usual embedding. Show that  $H = f^{-1}(S_{n-1})$  for some homomorphism  $f : G \rightarrow S_n$ . (Hint: Let  $G$  act on the cosets of  $H$ .)
81. Show that if  $\sigma = \rho\lambda \in S_{m+n}$  is the product of an  $m$ -cycle  $\rho$  and an  $n$ -cycle  $\lambda$ , with  $\rho$  and  $\lambda$  disjoint and  $m \neq n$ , then the centralizer in  $S_{m+n}$  of  $\sigma$  is  $\langle \rho, \lambda \rangle$ .
82. Let  $\tau$  be an element of the symmetric group  $S_n$  and let  $\sigma \in S_n$  be a transposition. Show that the number of cycles in the cycle decomposition of  $\sigma\tau$  is either one more or one less than the number of cycles in the cycle decomposition of  $\tau$ .
83. Show that if  $\sigma \in S_n$  is an  $(n-1)$ -cycle, where  $n \geq 3$ , then  $C(\sigma) = \langle \sigma \rangle$ .

## Infinite Groups

84. Let  $A$  and  $B$  be subgroups of the additive group of rationals  $\mathbb{Q}$ . Show that if  $A$  is isomorphic to  $B$  and  $f : A \rightarrow B$  is an isomorphism, then there exists  $q \in \mathbb{Q}$  such that  $f(x) = qx$  for all  $x \in A$ .
85. (a) Prove that the additive group of the rational numbers is not cyclic.  
(b) Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

86. If  $G$  is a finitely generated group and  $n$  is a positive integer, prove that there are at most finitely many subgroups of index  $n$  in  $G$ . (HINT: Consider maps into the symmetric group  $S_n$ .)
87. Let  $G$  be a group with a proper subgroup of finite index. Show that  $G$  has a proper normal subgroup of finite index.
88. Let  $\mathbb{Q}$  be the additive group of rationals and  $\mathbb{Z}$  its subgroup of integers. Prove the following.
- If  $n$  is a positive integer, then  $\mathbb{Q}/\mathbb{Z}$  has an element of order  $n$ .
  - If  $n$  is a positive integer, then  $\mathbb{Q}/\mathbb{Z}$  has a unique subgroup of order  $n$ .
  - Every finite subgroup of  $\mathbb{Q}/\mathbb{Z}$  is cyclic.
89. Let  $G$  have the presentation  $G = \langle a, b \mid a^2 = 1, a^{-1}bab = 1 \rangle$ . Prove that  $G$  is infinite but the commutator subgroup of  $G$  is of finite index in  $G$ .
90. Let  $N$  be a normal subgroup of  $G$  with the order of  $N$  finite. Prove there is a normal subgroup  $M$  of  $G$  such that  $[G : M]$  is finite and  $nm = mn$  for all  $n \in N$  and  $m \in M$ .
91. Let  $G$  be a finitely presented group in which there are fewer relations than generators. Prove that  $G$  is necessarily infinite.

### **$p$ -Groups**

92. Show that the center of a finite  $p$ -group is non-trivial.
93. Show that if  $P$  is a finite  $p$ -group and  $\langle 1 \rangle \neq N \trianglelefteq P$ , then  $N \cap Z(P) \neq \langle 1 \rangle$ .
94. Let  $P$  be a finite  $p$ -group and let  $H$  be a proper subgroup of  $P$ . Prove that  $H$  is a proper subgroup of its normalizer  $N_P(H)$ .
95. Show that a group of order  $p^2$ , where  $p$  is a prime, must be abelian.
96. Let  $p$  be a prime and let  $G$  be a non-abelian group of order  $p^3$ .
- Show that the center  $Z(G)$  of  $G$  and the commutator subgroup of  $G$  are equal and of order  $p$ .
  - Show that  $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ .
97. Let  $p$  be a prime and let  $G$  be a group of order  $p^n$  satisfying the following property:  
 (\*) If  $A$  and  $B$  are subgroups of  $G$  then  $A \leq B$  or  $B \leq A$ .  
 Prove that  $G$  is a cyclic group.
98. Let  $G$  be a finite group. Prove that  $G$  is a cyclic  $p$ -group, for some prime  $p$ , if and only if  $G$  has exactly one conjugacy class of maximal subgroups.
99. Let  $G$  be a finite  $p$ -group for some prime  $p$ . Show that if  $G$  is not cyclic, then  $G$  has at least  $p + 1$  maximal subgroups.
100. Let  $P$  be a finite  $p$ -group in which all the non-identity elements of the center  $Z(P)$  have order  $p$ . If  $\{Z_i(P)\}$  is the upper central series of  $P$ , prove that for every  $i$ , every non-identity element of  $Z_{i+1}(P)/Z_i(P)$  has order  $p$ .
101. Let  $P$  be a  $p$ -group satisfying  $[P : Z(P)] = p^n$ . Show that  $|P'| \leq p^{\frac{n(n-1)}{2}}$ .  
 (Hint: Use induction on  $n$ . Apply the inductive hypothesis to a maximal subgroup of  $P$ .)

102. Let  $G$  be a group of order 16 with an element  $g$  of order 4. Prove that the subgroup of  $G$  generated by  $g^2$  is normal in  $G$ .

### Group Actions

103. Show that if the center of a group  $G$  is of index  $n$  in  $G$ , then every conjugacy class of  $G$  has at most  $n$  elements.
104. Let  $G_n = \text{GL}_n(\mathbb{C})$  be the group of invertible  $n \times n$  matrices with complex entries and let  $M_n = \text{M}_n(\mathbb{C})$  be the set of all  $n \times n$  complex matrices.
- (a) Show that for  $g \in G_n$  and  $m \in M_n$ ,  $g \cdot m = gm g^{-1}$  defines a (left) action of  $G_n$  on  $M_n$ .
- (b) For  $n = 2$  and  $n = 3$ , find a complete set of orbit representatives.
105. Let  $G$  be a finite group acting on a set  $A$  and suppose that for any two ordered pairs  $(a_1, a_2)$  and  $(b_1, b_2)$  of elements of  $A$ , there is an element  $g \in G$  such that  $g \cdot a_i = b_i$  for  $i = 1, 2$ . Show that if  $|A| = n$ , then  $|G|$  is divisible by  $n(n - 1)$ . [Hint: Show that if  $a \in A$  then  $G_a$  acts transitively on  $A - \{a\}$ .]
106. Let  $G$  be a group acting on a set  $\Omega$ . Show that the following are equivalent.
- (i) The action is doubly transitive (i.e., for any two ordered pairs  $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$  of elements of  $\Omega$  with  $\alpha_1 \neq \beta_1$  and  $\alpha_2 \neq \beta_2$ , there is an element  $g$  in  $G$  such that  $g \cdot \alpha_1 = \alpha_2$  and  $g \cdot \beta_1 = \beta_2$ ).
- (ii) For all  $\alpha \in \Omega$ , the stabilizer  $G_\alpha$  acts transitively on  $\Omega - \{\alpha\}$ .
107. Let  $G$  be a group acting transitively on the set  $\Omega$ . Show that if  $\alpha \neq \beta$  are elements of  $\Omega$ , then  $G_\alpha G_\beta$  is a proper subset of  $G$ .
108. Let  $G$  be a group acting transitively on a set  $A$ . Show that if there is an element  $a \in A$  such that  $G_a = \{1\}$ , then  $G_b = \{1\}$  for all  $b \in A$ .
109. Let the group  $G$  act transitively on the set  $\Omega$ , and let  $N$  be a normal subgroup of  $G$ . Prove that  $G$  permutes the  $N$ -orbits of  $\Omega$  and that these orbits all have the same size.
110. Let  $G$  act on a set  $A$  and let  $B$  be a subset of  $A$ . For  $g \in G$ , let  $g \cdot B = \{g \cdot b : b \in B\}$ . Show that  $H = \{g \in G : g \cdot B = B\}$  is a subgroup of  $G$ .
111. Let  $G$  be a group acting on the set  $S$  and let  $H$  be a subgroup of  $G$  acting transitively on  $S$ . Show that if  $t \in S$  then  $G = G_t H$ , where  $G_t$  is the stabilizer of  $t$  in  $G$ .
112. Let  $G$  be a finite group. Show that if  $G$  has a normal subgroup  $N$  of order 3 that is not contained in the center of  $G$ , then  $G$  has a subgroup of index 2. [Hint: The group  $G$  acts on  $N$  by conjugation.]
113. (a) Let  $G$  be a finite group acting on the finite set  $S$ . For  $g \in G$ , let

$$F(g) = |\{x \in S : g \cdot x = x\}|.$$

Show that the number of orbits is  $\frac{1}{|G|} \sum_{g \in G} F(g)$ .

- (b) Show that the number of conjugacy classes of a finite group  $G$  is  $\frac{1}{|G|} \sum_{g \in G} |C_G(g)|$ .

114. Let  $G$  be a subgroup of  $S_n$  that acts transitively on  $\{1, 2, \dots, n\}$ .
- Show that if  $G_1 = \{g \in G \mid g \cdot 1 = 1\}$  then  $[G : G_1] = n$ .
  - Show that if  $G$  is abelian then  $G$  is of order  $n$ .
115. Let  $G$  be a finite group acting transitively on a set  $\Omega$ . Fix  $\alpha \in \Omega$  and let  $G_\alpha$  be the stabilizer of  $\alpha$  in  $G$ . Let  $\Delta$  be the set of points fixed by  $G_\alpha$ , i.e.  $\Delta = \{\beta \in \Omega \mid \beta \cdot x = \beta \forall x \in G_\alpha\}$ . Show that  $\Delta$  is stabilized by  $N_G(G_\alpha)$  and that  $N_G(G_\alpha)$  acts transitively on  $\Delta$ .
116. Let  $G$  act transitively on a set  $\Omega$ , fix  $\alpha \in \Omega$ , and let  $H = G_\alpha$ . Show that the orbits of  $H$  on  $\Omega$  are in one-to-one correspondence with the  $H - H$  double cosets in  $G$ .
117. Let  $G$  act on a set  $\Omega$  and assume  $N$  is a normal subgroup of  $G$  that is contained in the kernel of the action. Show that there is a natural action of  $G/N$  on  $\Omega$  which satisfies the property that  $G$  is transitive if and only if  $G/N$  is transitive.
118. Let  $G$  be a group with a subgroup  $H$  of finite index  $n$ . Show that there is a homomorphism  $\varphi : G \rightarrow S_n$  with  $\ker \varphi \subseteq H$ .
119. Suppose a group  $G$  has a subgroup  $H$  with  $|G : H| = n < \infty$ . Prove that  $G$  has a normal subgroup  $N$  with  $N \subseteq H$  and  $|G : N| \leq n!$ .
120. Let  $n > 1$  be a fixed integer. Prove that there are only finitely many simple groups (up to isomorphism) containing a proper subgroup of index less than or equal to  $n$ .
121. Let  $n = p^m r$  where  $p$  is prime and  $r$  is an integer greater than 1 such that  $p$  does not divide  $r$ . Show that if there is a simple group of order  $n$ , then  $p^m$  divides  $(r - 1)!$ .
122. Show that if  $G$  is a simple group of order greater than 60, then  $G$  has no proper subgroup of index less than or equal to 5.
123. Let  $G$  be a finite simple group containing an element of order 21. Show that every proper subgroup of  $G$  has index at least 10.
124. Let  $G$  be a finite group and let  $K$  be a subgroup of index  $p$ , where  $p$  is the smallest prime dividing the order of  $G$ . Show that  $K$  is a normal subgroup of  $G$ .
125. Let  $G$  be a nonabelian finite simple group and let  $H$  be a subgroup of index  $p$ , where  $p$  is a prime. Prove that the number of distinct conjugates of  $H$  in  $G$  is  $p$ .
126. Let  $G$  be a finite simple group with a subgroup  $H$  of prime index  $p$ . Show that  $p$  must be the largest prime dividing the order of  $G$ .
127. Let  $G$  be a finite simple group and  $p$  a prime such that  $p^2$  divides the order of  $G$ . Show that  $G$  has no subgroup of index  $p$ .
128. Let  $G$  be a finite group in which a Sylow 2-subgroup is cyclic. Prove that there exists a normal subgroup  $N$  of odd order such that the index  $[G : N]$  is a power of 2. [Hint: Generalize the previous problem.]

129. (a) Let  $G$  be a subgroup of the symmetric group  $S_n$ . Show that if  $G$  contains an odd permutation then  $G \cap A_n$  is of index 2 in  $G$ .
- (b) Let  $G$  be a group of order  $2r$ , where  $r > 1$  is an odd integer. Show that in the regular permutation representation of  $G$ , an element  $t$  of  $G$  of order 2 corresponds to an odd permutation.
- (c) Show that a group of order  $2r$ , with  $r > 1$  an odd integer, cannot be simple.
130. Let  $G$  be a finite cyclic group and  $H$  a subgroup of index  $p$ ,  $p$  a prime. Suppose  $G$  acts on a set  $S$  and the restriction of the action to  $H$  is transitive. Let  $G_x, H_x$  be the stabilizer of  $x \in S$  in  $G, H$ , respectively. Show the following.
- (a)  $H_x = G_x \cap H$
- (b)  $[H : H_x] = [G : G_x] = |S|$
- (c)  $|S|$  is not divisible by  $p$ .
131. Let  $G$  be a finite group and  $p$  a prime. Then  $G$  acts on  $\text{Syl}_p(G)$  by conjugation; let  $\rho : G \rightarrow \text{Sym}(\text{Syl}_p(G))$  be the homomorphism corresponding to this action.
- (a)  $\rho(P)$  fixes exactly one point (element of  $\text{Syl}_p(G)$ ).
- (b) If  $P \in \text{Syl}_p(G)$  has order  $p$ , then  $\rho(x)$  is a product of one 1-cycle and a certain number of  $p$ -cycles, for  $x \in P - \{1\}$ .
- (c) If  $P \in \text{Syl}_p(G)$  has order  $p$  and  $y \in N_G(P) - C_G(P)$  then  $\rho(y)$  fixes at most  $r$  points, where  $r$  is the number of orbits under the action of  $\rho(P)$  (including the fixed point of part (a)).
132. Let  $G$  be a finite group acting faithfully and transitively on a set  $\Omega$ . Assume that there exists a normal subgroup  $N$  such that  $N$  acts regularly on  $\Omega$  (i.e.,  $G = G_\alpha N$  and  $G_\alpha \cap N = 1$  for all  $\alpha \in \Omega$ ). Prove that  $G_\alpha$  embeds as a subgroup of  $\text{Aut}(N)$ .

## Sylow Theorems

133. (a) Let  $G$  be a finite  $p$ -group acting on the finite set  $S$ . Let  $S_0$  be the set of all elements of  $S$  fixed by  $G$ . Show that  $|S| \equiv |S_0| \pmod{p}$ .
- (b) Show that if  $H$  is a  $p$ -subgroup of a finite group  $G$ , then  $[N_G(H) : H] \equiv [G : H] \pmod{p}$ .
- (c) State and prove Sylow's theorems.
134. Let  $G$  be a finite group and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Prove the following.
- (a) If  $M$  is any normal  $p$ -subgroup of  $G$  then  $M \leq P$ .
- (b) There is a normal  $p$ -subgroup  $N$  of  $G$  that contains all normal  $p$ -subgroups of  $G$ .
135. Let  $n$  be an integer and  $p$  a prime dividing  $n$ . Assume that there exists exactly one divisor  $d$  of  $n$  satisfying both  $d > 1$  and  $d \equiv 1 \pmod{p}$ . Prove that if  $G$  is any finite group of order  $n$  and  $P$  is a Sylow  $p$ -subgroup of  $G$ , then either  $P \trianglelefteq G$  or else  $N_G(P)$  is a maximal subgroup of  $G$ .
136. Let  $G$  be a group of order 168 and let  $P$  be a Sylow 7-subgroup of  $G$ . Show that either  $P$  is a normal subgroup of  $G$  or else the normalizer of  $P$  is a maximal subgroup of  $G$ .
137. Show that if  $G$  is a simple group of order 60 then  $G \cong A_5$ .

138. Show that a group of order  $2001 = 3 \cdot 23 \cdot 29$  contains a normal cyclic subgroup of index 3.
139. Show that if  $G$  is a group of order  $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ , then  $G$  has an abelian subgroup of index 2.
140. Show that a group of order  $2004 = 2^2 \cdot 3 \cdot 167$  must be solvable. Give an example of a group of order 2004 in which a Sylow 3-subgroup is not a normal subgroup.
141. Show that if  $G$  is a group of order  $2010 = 2 \cdot 3 \cdot 5 \cdot 67$ , then  $G$  has a normal subgroup of order 5.
142. Prove that a group  $G$  of order 36 must have a normal subgroup of order 3 or 9.
143. Show that a group of order 96 must have a normal subgroup of order 16 or 32.
144. Show that a group of order  $160 = 2^5 \cdot 5$  must contain a nontrivial normal 2-subgroup.
145. Show that if  $G$  is a group of order  $392 = 2^3 \cdot 7^2$ , then  $G$  has a normal subgroup of order 7 or a normal subgroup of order 49.
146. Let  $G$  be a finite simple group containing an element of order 9. Show that every proper subgroup of  $G$  has index at least 9.
147. Show that there is no simple group of order 120.
148. (a) Show that  $S_6$  has no simple subgroup of index 4 (i.e. order 180).  
 (b) Show that a group of order  $180 = 2^2 \cdot 3^2 \cdot 5$  cannot be simple.
149. (a) Show that  $|\text{Aut}(\mathbb{Z}_7)| = 6$ .  
 (b) Show that a group of order 63 must contain an element of order 21.
150. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group  $A_8$ .
151. Let  $G$  be a simple group of order 168. Determine the number of elements of  $G$  of order 7. Explain your answer.
152. Let  $p > q$  be primes. Show that if  $p - 1$  is not divisible by  $q$ , then there is exactly one group of order  $pq$ .
153. Let  $G$  be a group of order  $pqr$ , where  $p > q > r$  are primes. Let  $P$  be a Sylow  $p$ -subgroup of  $G$  and assume  $P$  is not normal in  $G$ . Show that a Sylow  $q$ -subgroup of  $G$  must be normal.
154. Let  $G$  be a group of order  $pqr$ , where  $p > q > r$  are primes. Show that if  $p - 1$  is not divisible by  $q$ , then a Sylow  $p$ -subgroup of  $G$  must be normal.
155. Let  $G$  be a group of order  $pqr$ , where  $p > q > r$  are primes. Show that if  $p - 1$  is not divisible by  $q$  or  $r$  and  $q - 1$  is not divisible by  $r$ , then  $G$  must be abelian (hence cyclic).  
 [Hint: Show that  $G'$  must be contained in a Sylow subgroup for two different primes.]
156. Let  $G$  be a group of order  $105 = 3 \cdot 5 \cdot 7$ . Prove that a Sylow 7-subgroup of  $G$  is normal.
157. Show that a group  $G$  of order  $255 = 3 \cdot 5 \cdot 17$  must be abelian.

158. Let  $G$  be a group of order  $231 = 3 \cdot 7 \cdot 11$ . Prove that a Sylow 11-subgroup is contained in the center of  $G$ .
159. Show that a group of order  $10000 = 2^4 \cdot 5^4$  cannot be simple.
160. Show that a group of order  $3^3 \cdot 5 \cdot 13$  must have a normal Sylow 13-subgroup or a normal Sylow 5-subgroup. [Hint: Show that if a Sylow 13-subgroup is not normal, then a Sylow 13-subgroup must normalize a Sylow 5-subgroup. Consider the normalizer of a Sylow 5-subgroup.]
161. Let  $G$  be a group of order  $3 \cdot 5 \cdot 7 \cdot 13$ . Prove that  $G$  is not a simple group. [Hint: If a Sylow 7-subgroup is not normal, then some Sylow 13-subgroup will centralize it. Now compute the number of Sylow 13-subgroups.]
162. Let  $G$  be a group of order  $p^n q$ , where  $p$  and  $q$  are distinct primes, and assume  $q \nmid p^i - 1$  for  $1 \leq i \leq n - 1$ . Prove that  $G$  is solvable.
163. Let  $p$  and  $q$  be distinct primes. Show that a group of order  $p^2 q$  has a normal Sylow  $p$ -subgroup or a normal Sylow  $q$ -subgroup.
164. Let  $G$  be a group of order  $(p + 1)p(p - 1)$  where  $p$  is a prime. Prove that the number of Sylow  $p$ -subgroups is either 1 or  $p + 1$ .
165. Show that a group of order  $2^3 \cdot 3 \cdot 7^2$  is not simple.
166. Show that a group of order  $380 = 2^2 \cdot 5 \cdot 19$  must be solvable.
167. Show that a group of order  $2 \cdot 7 \cdot 13$  must be solvable.
168. Show that a group of order  $1960 = 2^3 \cdot 5 \cdot 7^2$  must be solvable.
169. Prove that a group of order  $1995 = 3 \cdot 5 \cdot 7 \cdot 19$  must be solvable.
170. Show that a group of order  $1998 = 2 \cdot 3^3 \cdot 37$  must be solvable.
171. Let  $G$  be a group with exactly 31 Sylow 3-subgroups. Prove that there exist Sylow 3-subgroups  $P$  and  $Q$  satisfying  $[P : P \cap Q] = [Q : P \cap Q] = 3$ .
172. Let  $G$  be a finite group,  $p$  a prime divisor of  $|G|$  and assume there are  $k$  distinct Sylow  $p$ -subgroups of  $G$ . Let  $f : G \rightarrow S_k$  be the homomorphism of  $G$  into the symmetric group induced by the natural action of  $G$  by conjugation on the set of Sylow  $p$ -subgroups of  $G$ , and let  $\overline{G} = f(G)$ . Prove that  $\overline{G}$  has  $k$  distinct Sylow  $p$ -subgroups.
173. (a) Show that if  $K$  is a subgroup of  $G$  then the number of distinct conjugates of  $K$  in  $G$  is  $[G : N_G(K)]$ .
- (b) Show that if  $G$  has  $n_p$  Sylow  $p$ -subgroups, then  $G$  has a subgroup of index  $n_p$ .
174. Let  $G$  be a finite group and  $p$  a prime. Show that the intersection of all Sylow  $p$ -subgroups of  $G$  is a normal subgroup of  $G$ .
175. Let  $K$  be a normal subgroup of  $G$  and let  $P$  be a Sylow  $p$ -subgroup of  $K$ . Show that if  $P \trianglelefteq K$  then  $P \trianglelefteq G$ .
176. Let  $G$  be a finite group and let  $P$  be a *normal* Sylow  $p$ -subgroup of  $G$ . Show that  $P$  is a characteristic subgroup of  $G$ .

177. A subgroup  $H$  of a group  $G$  is subnormal if there exists a chain  $H = H_0 \leq H_1 \leq \cdots \leq H_k = G$  such that  $H_i$  is a normal subgroup of  $H_{i+1}$  for every  $i$ . Prove that if  $P$  is a Sylow  $p$ -subgroup of a finite group  $G$  then  $P$  is a subnormal in  $G$  if and only if  $P$  is normal in  $G$ .
178. Let  $G$  be a finite group and  $p$  a prime. Let  $N$  be a normal subgroup of  $G$  and  $H$  a Sylow  $p$ -subgroup of  $G$ . Show that
- $HN/N$  is a Sylow  $p$ -subgroup of  $G/N$ , and
  - $H \cap N$  is a Sylow  $p$ -subgroup of  $N$ .
179. Let  $G$  be a finite group with subgroups  $H, K$  such that  $G = HK$ . Show that if  $p$  is any prime number, then there exist  $P \in \text{Syl}_p(H)$  and  $Q \in \text{Syl}_p(K)$  such that  $PQ \in \text{Syl}_p(G)$ .
180. Let  $G$  be a finite group,  $p$  a prime, and  $P$  a Sylow  $p$ -subgroup of  $G$ . Let  $H$  be a subgroup of  $G$  that contains the normalizer  $N_G(P)$  of  $P$  in  $G$ . Show that if  $g$  is an element of  $G$  such that  $g^{-1}Pg \leq H$ , then  $g$  is an element of  $H$ .
181. Let  $G$  be a finite group,  $H$  be a subgroup of  $G$ , and  $P$  be a Sylow  $p$ -subgroup of  $H$  for some prime  $p$ . Show that if  $H$  contains the normalizer  $N_G(P)$  of  $P$ , then  $P$  is a Sylow  $p$ -subgroup of  $G$ .
182. A subgroup  $H$  of a group  $G$  is called *pronormal* if, for any  $g \in G$ ,  $H$  is conjugate to  $H^g$  in  $\langle H, H^g \rangle$ .
- Show that if  $H \leq N \trianglelefteq G$  with  $H$  pronormal in  $G$ , then  $G = N_G(H)N$ .
  - Show that if  $P$  is a Sylow  $p$ -subgroup of  $G$ , then  $P$  is pronormal in  $G$ .
183. Let  $G$  be a finite group and  $H$  a normal subgroup. Show that if  $P$  is a Sylow  $p$ -subgroup of  $H$ , then  $G = HN_G(P)$ .
184. Let  $P$  be a Sylow  $p$ -subgroup of a group  $G$  and let  $K$  be a subgroup of  $G$  containing  $N_G(P)$ . Show that  $N_G(K) = K$ .
185. Let  $x$  and  $y$  be two elements of  $Z(P)$  where  $P$  is a Sylow  $p$ -subgroup of  $G$ . If  $x$  and  $y$  are conjugate in  $G$ , prove that  $x$  is conjugate to  $y$  in  $N_G(P)$ .
186. (a) Let  $p$  be a prime and let  $H$  be a  $p$ -subgroup of the finite group  $G$ . Show that
- $$[N_G(H) : H] \equiv [G : H] \pmod{p}.$$
- (Hint: Let  $H$  act on  $G/H$  by left multiplication.)
- Let  $P$  be a  $p$ -subgroup of  $G$ . Show that  $P$  is a Sylow  $p$ -subgroup of  $G$  if and only if  $P$  is a Sylow  $p$ -subgroup of  $N_G(P)$ .
187. Let  $G$  be a finite group with  $|G| = p^a m$ , where  $p$  is a prime and  $p \nmid m$ . Assume that whenever  $P$  and  $Q$  are Sylow  $p$ -subgroups of  $G$ , either  $P = Q$  or  $P \cap Q = 1$ . Show that the number of Sylow  $p$ -subgroups of  $G$  is congruent to 1 modulo  $p^a$ .
188. Let  $P$  be a Sylow  $p$ -subgroup of the finite group  $G$ , and assume  $|P| = p^a$ . Suppose that  $P \cap P^g = \{1\}$  whenever  $g \in G$  does not normalize  $P$ . Prove that the number of Sylow  $p$ -subgroups of  $G$  is congruent to 1 mod  $p^a$ .

189. Let  $p$  be a prime and let  $P$  be a  $p$ -subgroup of the finite group  $G$ . Show that  $P$  is a Sylow  $p$ -subgroup of  $G$  if and only if  $P$  is a Sylow  $p$ -subgroup of  $PC_G(P)$  and  $[N_G(P) : PC_G(P)]$  is not divisible by  $p$ .
190. Let  $G$  be a finite group, and  $p$  a prime. Let  $n_p$  be the number of Sylow  $p$ -subgroups of  $G$  and suppose that  $p^e$  does *not* divide  $n_p - 1$ . Prove that there exist two distinct Sylow  $p$ -subgroups of  $G$ , say  $P$  and  $Q$ , satisfying  $[P : P \cap Q] \leq p^e$ .
191. Let  $P$  and  $Q$  be distinct Sylow  $p$ -subgroups of a finite group  $G$ . Prove that the number of Sylow  $p$ -subgroups of  $G$  is strictly greater than  $[P : P \cap Q]$ .
192. Let  $X$  and  $G$  be finite groups. We say that  $X$  is *involved* in  $G$  if there exist subgroups  $K$  and  $H$  of  $G$ , with  $K$  normal in  $H$ , such that  $X$  is isomorphic to  $H/K$ . Suppose  $X$  is a  $p$ -group,  $P$  is a Sylow  $p$ -subgroup of  $G$ , and  $X$  is involved in  $G$ . Prove that  $X$  is involved in  $P$ .

### Solvable and Nilpotent Groups, Commutator and Frattini Subgroups

193. Show that the following statements are equivalent.
- (i) Every finite group of odd order is solvable.
  - (ii) Every non-abelian finite simple group is of even order.
194. Let  $H$  and  $K$  be subgroups of a group  $G$  with  $K \trianglelefteq G$ . Show that if  $H$  and  $K$  are solvable, then  $HK$  is solvable.
195. Let  $G$  be a solvable group and  $N$  a nontrivial normal subgroup of  $G$ . Show that there is a nontrivial abelian subgroup  $A$  of  $N$  with  $A$  normal in  $G$ .
196. Let  $G$  be a finite non-solvable group, each of whose proper subgroups is solvable. Show that  $G/\Phi(G)$  is a non-abelian simple group, where  $\Phi(G)$  denotes the Frattini subgroup of  $G$ .
197. We say that a group  $X$  is *involved* in a group  $G$  if  $X$  is isomorphic to  $H/K$  for some subgroups  $K, H$  of  $G$  with  $K \trianglelefteq H$ . Prove that if  $X$  is solvable and  $X$  is involved in the finite group  $G$ , then  $X$  is involved in a solvable subgroup of  $G$ .
198. Let  $G$  be a finite group satisfying the following property:  
 (\*) If  $A, B$  are subgroups of  $G$  then  $AB$  is a subgroup of  $G$ .  
 Prove that  $G$  is a solvable group.
199. Let  $X$  be a set of operators for the group  $G$  and assume that  $G$  is a finite solvable group. Prove that every  $X$ -composition factor in any  $X$ -composition series for  $G$  is an elementary abelian  $p$ -group for some prime  $p$ .
200. Show that if  $G$  is a nilpotent group and  $\langle 1 \rangle \neq N \trianglelefteq G$ , then  $N \cap Z(G) \neq \langle 1 \rangle$ .
201. Show that if  $G$  is a nilpotent finite group, then every subgroup of prime index is a normal subgroup.

202. (a) Show that if  $G$  is a group and  $H, K$  are subgroups of  $G$  such that  $HK \subseteq KH$ , then  $HK$  is a subgroup of  $G$ .
- (b) Suppose  $G$  is finite and  $HK \subseteq KH$  for all subgroups  $H$  and  $K$  of  $G$ . Show that if  $p$  is a prime divisor of  $|G|$ , then there is a subgroup  $N$  of  $G$  such that  $|G : N|$  is a power of  $p$  and  $p \nmid |N|$ .
203. Let  $G$  be a finite group and let  $\Phi(G)$  be its Frattini subgroup. Show that  $\Phi(G)$  is precisely the set of non-generators of  $G$ . (An element  $g$  of  $G$  is called a non-generator if for any subset  $S$  of  $G$  containing  $g$  and generating  $G$ , the set  $S - \{g\}$  also generates  $G$ .)
204. Let  $\langle 1 \rangle = G_0 \leq G_1 \leq \dots \leq G_n = G$  be a central series for the nilpotent group  $G$ . Prove that  $G_i \leq Z_i(G)$  for all  $i$ , where  $\{Z_i(G)\}$  is the upper central series of  $G$ . Thus, among all central series for a nilpotent group, the upper central series ascends the fastest.
205. Let  $G$  be a finite group, let  $\Phi(G)$  be the Frattini subgroup of  $G$  (that is, the intersection of all maximal subgroups of  $G$ ) and let  $G'$  be the commutator subgroup of  $G$ . Show that the following are equivalent.
- The group  $G$  is nilpotent.
  - If  $H$  is a proper subgroup of  $G$ , then  $H$  is a proper subgroup of its normalizer in  $G$ .
  - Every maximal subgroup of  $G$  is a normal subgroup of  $G$ .
  - $G' \leq \Phi(G)$ .
  - Every Sylow subgroup of  $G$  is a normal subgroup of  $G$ .
  - The group  $G$  is a direct product of its Sylow subgroups.
206. Let  $G$  be a finite group. Show that each of the following conditions is equivalent to the nilpotence of  $G$ .
- Whenever  $x, y \in G$  satisfy  $(|x|, |y|) = 1$ , then  $xy = yx$ .
  - Whenever  $p$  and  $q$  are distinct primes and  $P \in \text{Syl}_p(G)$  and  $Q \in \text{Syl}_q(G)$ , then  $P$  centralizes  $Q$ .
207. Show that if  $G$  is a finite nilpotent group and  $m$  is a positive integer such that  $m$  divides the order of  $G$ , then  $G$  has a subgroup of order  $m$ .
208. Let  $G$  be a finite nilpotent group and  $G'$  its commutator subgroup. Show that if  $G/G'$  is cyclic then  $G$  is cyclic.
209. A finite group  $G$  is called an  $N$ -group if the normalizer  $N_G(P)$  of every non-identity  $p$ -subgroup  $P$  of  $G$  is solvable. Prove that if  $G$  is an  $N$ -group, then either (i)  $G$  is solvable, or (ii)  $G$  has a unique minimal normal subgroup  $K$ , the factor group  $G/K$  is solvable, and  $K$  is simple.
210. Let  $G$  be a finite group and let  $N$  be a normal subgroup of  $G$  with the property that  $G/N$  is nilpotent. Prove that there exists a nilpotent subgroup  $H$  of  $G$  satisfying  $G = HN$ .
211. Let  $G$  be a finite solvable group. Prove that the index of every maximal subgroup is a prime power.
212. **[NEW]**  
Let  $G$  be a group. Show that if  $g \in G$ , then the conjugacy class of  $g$  is contained in  $gG'$ .

213. Let  $G$  be a group of odd order. Let  $g_1, \dots, g_n$  be the elements of  $G$ , listed in any order.

Show that  $\prod_{i=1}^n g_i$  is an element of the commutator subgroup  $G'$  of  $G$ .

214. Let  $G$  be a finite group and let  $M$  be a maximal subgroup of  $G$ .

(a) Show that if  $Z(G)$  is not contained in  $M$ , then  $M \trianglelefteq G$ .

(b) Show that either  $Z(G) \leq M$  or  $G' \leq M$ .

(c) Show that  $Z(G) \cap G' \leq \Phi(G)$ .