ALGEBRA QUALIFYING EXAM PROBLEMS
LINEAR ALGEBRA

Kent State University
Department of Mathematical Sciences

Compiled and Maintained
by
Donald L. White

Version: May 27, 2009

CONTENTS

LINEAR ALGEBRA

General Matrix Theory

1. Let $m > n$ be positive integers. Show that there do not exist matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ such that $AB = I_m$, where I_m is the $m \times m$ identity matrix.

2. Let A and B be nonsingular $n \times n$ matrices over $\mathbb{C}$.
   (a) Show that if $A^{-1}B^{-1}AB = cI$, $c \in \mathbb{C}$, then $c^n = 1$.
   (b) Show that if $AB - BA = cI$, $c \in \mathbb{C}$, then $c = 0$.

3. Let A be a strictly upper triangular $n \times n$ matrix with real entries, and let I be the $n \times n$ identity matrix. Show that $I - A$ is invertible and express the inverse of $I - A$ as a function of A.

4. (a) Give an example of a complex 2×2 matrix which does not have a square root.
   (b) Show that every complex non-singular $n \times n$ matrix has a square root. [Hint: Show first that a Jordan block with non-zero eigenvalue has a square root.]

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $T(v) = Av$, where $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}$.
   (a) Find the dimension of the null space of T.
   (b) Find a basis for the range space of T.

Canonical Forms, Diagonalization, and Characteristic and Minimal Polynomials

6. State and prove the Cayley-Hamilton Theorem.

7. Show that if A is an $n \times n$ matrix, then A^n can be written as a linear combination of the matrices $I, A, A^2, \ldots, A^{n-1}$ (that is, $A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{n-1}A^{n-1}$ for some scalars $\alpha_0, \ldots, \alpha_{n-1}$).

8. Let A be an $n \times n$ Jordan block. Show that any matrix that commutes with A is a polynomial in A.

9. Let A be a square matrix whose minimal polynomial is equal to its characteristic polynomial. Show that if B is any matrix that commutes with A, then B is a polynomial in A.

10. Prove that an $n \times n$ complex matrix A is diagonalizable if and only if the minimal polynomial of A has distinct roots.

11. Let $G = \mathrm{GL}_n(\mathbb{C})$ be the multiplicative group of invertible $n \times n$ matrices with complex entries and let g be an element of G of finite order. Show that g is diagonalizable.

12. Let V be a vector space and let $T : V \to V$ be a linear transformation.
    (a) Show that T is invertible if and only if the minimal polynomial of T has non-zero constant term.
    (b) Show that if T is invertible then T^{-1} can be expressed as a polynomial in T.

13. Let A and B be 5×5 complex matrices and suppose that A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x+1)^2$ and the characteristic polynomial of B is x^5, then $B^3 = 0$.

14. A square matrix A over $\mathbb{C}$ is *Hermitian* if $\bar{A}^t = A$. Prove that the eigenvalues of a Hermitian matrix are all real.

15. (a) Prove that a 2×2 scalar matrix A over a field F has a square root (i.e., a matrix B satisfying $B^2 = A$).

    (b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root. [Hint: Use (a).]

16. Let A and B be complex $n \times n$ matrices. Prove that if $AB = BA$, then A and B share a common eigenvector.

17. Let A be a 5×5 matrix with complex entries such that $A^3 = 0$. Find all possible Jordan Canonical Forms for A.

18. (a) Show that two 3×3 complex matrices are similar if and only if they have the same characteristic and minimal polynomials.

    (b) Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.

19. (a) Find all possible Jordan canonical forms of a 5×5 complex matrix with minimal polynomial $(x-2)^2(x-1)$.

    (b) Find all possible Jordan canonical forms of a complex matrix with characteristic polynomial $(x-3)^3(x-5)^2$.

20. Find all possible Jordan canonical forms for the following. EXPLAIN your answers.

    (a) A linear operator T with characteristic polynomial $\Delta(x) = (x-2)^4(x-3)^2$ and minimal polynomial $m(x) = (x-2)^2(x-3)^2$.

    (b) A linear operator T with characteristic polynomial $\Delta(x) = (x-4)^5$ and such that $\dim \ker(T - 4I) = 3$.

21. A matrix A has characteristic polynomial $\Delta(x) = (x-3)^5$ and minimal polynomial $m(x) = (x-3)^3$.

    (a) List all possible Jordan canonical forms for A.

    (b) Determine the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 & 0 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 2 & -3 & 0 & 3 \end{bmatrix}$$

    which has the given characteristic and minimal polynomials.

22. Let $T : V \longrightarrow V$ be a linear transformation defined on the finite dimensional vector space V. Let λ be an eigenvalue of T, and set $W_i = \{v \in V \mid (T - \lambda I)^i(v) = 0\}$. If m is the multiplicity of λ as a root of the characteristic polynomial of T, prove that $W_m = W_{m+1}$.

23. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ be a matrix over the field F, where F is either the field of rational numbers or the field of p elements for some prime p.

(a) Find a basis of eigenvectors for A over those fields for which such a basis exists.

(b) What is the Jordan canonical form of A over the fields not included in part (a)?

24. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, let $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 3 \end{bmatrix}$, and let $C = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial of A, B, and C.

(b) Find the minimal polynomial of A, B, and C.

(c) Find the eigenvalues of A, B, and C.

(d) Find the dimensions of all eigenspaces of A, B, and C.

(e) Find the Jordan canonical form of A, B, and C.

25. Let $A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$

(a) Determine conditions on a, b, c, and d so that there is only one Jordan block for each eigenvalue of A in the Jordan canonical form of A.

(b) Suppose now $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

26. Let A be a square complex matrix with a single eigenvalue λ. Show that the number of blocks in the Jordan form of A is the dimension of the λ-eigenspace.

27. Let A be an $n \times n$ nilpotent matrix such that $A^{n-1} \neq 0$. Show that A has exactly one Jordan block.

28. Let $A = \begin{bmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{bmatrix}$ with characteristic polynomial $\Delta(x) = (x-1)^2(x-2)$.

(a) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

(b) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

29. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

(a) Verify that the characteristic polynomial of A is $\Delta(x) = x(x-1)^2$.

(b) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

(c) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

30. Let $A = \begin{bmatrix} 5 & 0 & 6 \\ 2 & 2 & 4 \\ -2 & 0 & -2 \end{bmatrix}$.

   (a) Verify that the characteristic polynomial of A is $\Delta(x) = (x-1)(x-2)^2$.

   (b) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

   (c) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

31. Let A be a matrix of the form $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}$.
   Show that the minimal polynomial and characteristic polynomial of A are equal.

Linear Transformations

32. (Fitting's Lemma for vector spaces) Let $\varphi : V \to V$ be a linear transformation of a finite dimensional vector space to itself. Prove that there exists a decomposition of V as $V = U \oplus W$ where each summand is φ-invariant, $\varphi|_U$ is nilpotent, and $\varphi|_W$ is nonsingular.

33. Let V be a vector space over a field F. A linear transformation $T : V \to V$ is said to be *idempotent* if $T^2 = T$. Prove that if T is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.

34. Let U, V and W be finite dimensional vector spaces with U a subspace of V. Show that if $T : V \longrightarrow W$ is a linear transformation having the same rank as $T|_U : U \longrightarrow W$, then U is complemented in V by a subspace K satisfying $T(x) = 0$ for all $x \in K$.

35. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\operatorname{Im} T) = \dim(V)$.

36. Let V be a finite dimensional vector space and $T : V \to V$ a *non-zero* linear operator. Show that if $\ker T = \operatorname{Im} T$, then $\dim V$ is an *even* integer and the minimal polynomial of T is $m(x) = x^2$.

37. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nilpotent linear transformation. Show that the trace of T is 0.

38. Let $T : V \to W$ be a surjective linear transformation of finite dimensional vector spaces over a field F (acting on the left). Show that there is a linear transformation $S : W \to V$ such that $T \circ S$ is the identity map on W.

39. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one.

40. Let $T : V \to W$ be a linear transformation of vector spaces over a field F.
   (a) Show that T is injective if and only if $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent in W whenever $\{v_1, \ldots, v_n\}$ is linearly independent in V.
   (b) Show that T is surjective if and only if $\{T(x) \mid x \in X\}$ is a spanning set for W whenever X is a spanning set for V.

41. Let A be a complex $n \times n$ matrix, and let $L : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$ be the linear transformation given by $L(M) = AM$ for $M \in \mathbb{C}^{n\times n}$. Express det L in terms of det A and prove your formula is correct.

42. Let A be a complex $n \times n$ matrix, and let $L : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$ be the linear transformation given by $L(X) = AX + XA$ for $X \in \mathbb{C}^{n\times n}$. Prove that if A is a nilpotent matrix, then L is a nilpotent operator.

43. Let $T : V \to V$ be a linear transformation. Let $X = \ker T^{n-2}$, $Y = \ker T^{n-1}$, and $Z = \ker T^n$. Observe that $X \subseteq Y \subseteq Z$ (you need not prove this). Suppose

$$\{u_1, \ldots, u_r\},\ \{u_1, \ldots, u_r, v_1, \ldots, v_s\},\ \{u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t\}$$

are bases for X, Y, Z, respectively. Show that $S = \{u_1, \ldots, u_r, T(w_1), \ldots, T(w_t)\}$ is contained in Y and is linearly independent.

Vector Spaces

44. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis for a vector space V over $\mathbb{R}$. Show that if $\boldsymbol{w}$ is any vector in V, then for some choice of sign $\pm$, $\{\boldsymbol{v}_1 \pm \boldsymbol{w}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for V.

45. Let V be a finite dimensional vector space over the field F. Let V^* be the dual space of V (that is, V^* is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$.

46. Let V be a vector space over the field F. Let V^* be the dual space of V and let V^{**} be the dual space of V^*. Show that there is an injective linear transformation $\varphi : V \to V^{**}$.

47. Let V be a finite dimensional vector space and let W be a subspace. Show that $\dim V/W = \dim V - \dim W$.

48. Let V be a vector space and let U and W be finite dimensional subspaces of V. Show that $\dim(U + W) = \dim U + \dim W - \dim U \cap W$.

49. Let V be a finite-dimensional vector space over a field F and let U be a subspace. Show that there is a subspace W of V such that $V = U \oplus W$.

50. Let V be the vector space of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. Let U be the subspace of V consisting of symmetric matrices and W the subspace of V consisting of skew-symmetric matrices. Show that $V = U \oplus W$.

51. Let V be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

52. Let U, V, and W be vector spaces over a field F and let $S : U \to V$ and $T : V \to W$ be linear transformations such that $T \circ S = \mathbf{0}$, the zero map. Show that

$$\dim(W/\text{Im } T) - \dim(\ker T/\text{Im } S) + \dim \ker S = \dim W - \dim V + \dim U.$$

53. Let A, B, and C be subspaces of the nonzero vector space V satisfying

$$V = A \oplus B = B \oplus C = A \oplus C.$$

Show that there exists a 2-dimensional subspace $W \subseteq V$ such that each of $W \cap A$, $W \cap B$, and $W \cap C$ has dimension 1.

54. Let

$$V_{-1} = 0 \xrightarrow{L_{-1}=0} V_0 \xrightarrow{L_0} V_1 \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} V_n \xrightarrow{L_n=0} V_{n+1} = 0$$

be a sequence of finite dimensional vector spaces and linear transformations with $L_{i+1} \circ L_i = 0$ for all $i = 0, \ldots, n$. Therefore, the quotients $H_i = \ker(L_i)/\text{im}(L_{i-1})$ are defined for $0 \leqslant i \leqslant n$. Prove that $\sum_i (-1)^i \dim V_i = \sum_i (-1)^i \dim H_i$.

55. If V is a finite dimensional vector space, let V^* denote the dual of V. If $(\cdot, \cdot)$ is a non-degenerate bilinear form on V, and W is a subspace of V, define $W^\perp = \{v \in V | (v, w) = 0 \text{ for all } w \in W\}$. Show that if X and Y are subspaces of V with $Y \subseteq X$, then $X^\perp \subseteq Y^\perp$ and $Y^\perp/X^\perp \cong (X/Y)^*$.

Inner Product Spaces

56. Let $(\ ,\ )$ be a positive definite inner product on the finite dimensional real vector space V. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors satisfying $(v_i, v_j) < 0$ for all $i \neq j$. Prove that $\dim(\text{span } S) \geqslant k - 1$.

57. Let $\{v_1, v_2, \ldots, v_k\}$ be a linearly independent set of vectors in the real inner product space V. Show that there exists a unique set $\{u_1, u_2, \ldots, u_k\}$ of vectors with the property that $(u_i, v_i) > 0$ for all i, and $\{u_1, u_2, \ldots, u_i\}$ is an orthonormal basis for $\text{Span}\{v_1, v_2, \ldots, v_i\}$ for every i.

58. Let $(\ ,\ )$ be a Hermitian inner product defined on the complex vector space V. If $\varphi : V \to V$ is a normal operator ($\varphi \circ \varphi^* = \varphi^* \circ \varphi$, where φ^* is the adjoint of φ), prove that V contains an orthonormal basis of eigenvectors for φ.

Modules

59. Let M be a nonzero R-module with the property that every R-submodule N is complemented (that is, there exists another R-submodule C such that $M = N + C$ and $N \cap C = \{0\}$). Give a direct proof that M contains simple submodules.

60. Let $R = F^{n\times n}$ be the ring of $n \times n$ matrices over a field F. Prove that the (right) R-module $F^{1\times n}$, consisting of the row space of $1 \times n$ matrices, is the unique simple R-module (up to isomorphism).

61. Let $R \subseteq S$ be an inclusion of rings (sharing the same identity element). Let S_R be the right R-module where the module action is right multiplication. Assume S_R is isomorphic to a direct sum of n copies of R. Prove that S is isomorphic to a subring of $M_n(R)$, the ring of $n \times n$ matrices over R.

62. Let M be a module over a ring R with identity, and assume that M has finite composition length. If $\varphi : M \to M$ is an R-endomorphism of M, prove that M decomposes as a direct sum of R-submodules $M = U \oplus W$ where each summand is φ-invariant, $\varphi|_U$ is nilpotent, and $\varphi|_W$ is an automorphism.

63. Let R be a ring with identity, and let I be a right ideal of R which is a direct summand of R (i.e., $R = I \oplus J$ for some right ideal J). Prove that if M is any R-module, and $\varphi : M \to I$ is any surjective R-homomorphism, then there exists an R-homomorphism $\psi : I \to M$ satisfying $\varphi \circ \psi = 1|_I$.

64. Let M be an R-module and let N be an R-submodule of M. Prove that M is Noetherian if and only if both N and M/N are Noetherian.

65. Let M be an R-module and let N be an R-submodule of M. Prove that M is Artinian if and only if both N and M/N are Artinian.

66. Let M be an R-module, where R is a ring. Prove that the following statements about M are equivalent.
    (i) M is a sum (not necessarily direct) of simple submodules.
    (ii) M is a direct sum of certain simple submodules.
    (iii) For every submodule N of M, there exists a complement (i.e., a submodule C such that $M = N + C$ and $N \cap C = 0$).

67. Let R be a ring and let M be a simple R-module. Let $D = \mathrm{End}_R(M)$ be the ring of R-endomorphisms of M (under composition and pointwise addition). Prove that D is a division ring.

68. Let M be an R-module that is generated by finitely many simple submodules. Prove that M is a direct sum of finitely many simple R-modules.