ALGEBRA QUALIFYING EXAM PROBLEMS
RING THEORY

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### RING THEORY

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RING THEORY

General Ring Theory

1. Give an example of each of the following.
   (a) An irreducible polynomial of degree 3 in \( \mathbb{Z}_3[x] \).
   (b) A polynomial in \( \mathbb{Z}[x] \) that is not irreducible in \( \mathbb{Z}[x] \) but is irreducible in \( \mathbb{Q}[x] \).
   (c) A non-commutative ring of characteristic \( p \), \( p \) a prime.
   (d) A ring with exactly 6 invertible elements.
   (e) An infinite non-commutative ring with only finitely many ideals.
   (f) An infinite non-commutative ring with non-zero characteristic.
   (g) An integral domain which is not a unique factorization domain.
   (h) A unique factorization domain that is not a principal ideal domain.
   (i) A principal ideal domain that is not a Euclidean domain.
   (j) A Euclidean domain other than the ring of integers or a field.
   (k) A finite non-commutative ring.
   (l) A commutative ring with a sequence \( \{P_n\}_{n=1}^{\infty} \) of prime ideals such that \( P_n \) is properly contained in \( P_{n+1} \) for all \( n \).
   (m) A non-zero prime ideal of a commutative ring that is not a maximal ideal.
   (n) An irreducible element of a commutative ring that is not a prime element.
   (o) An irreducible element of an integral domain that is not a prime element.
   (p) A commutative ring that has exactly one maximal ideal and is not a field.
   (q) A non-commutative ring with exactly two maximal ideals.

2. (a) How many units does the ring \( \mathbb{Z}/60\mathbb{Z} \) have? Explain your answer.
    (b) How many ideals does the ring \( \mathbb{Z}/60\mathbb{Z} \) have? Explain your answer.

3. [NEW] How many ideals does the ring \( \mathbb{Z}/90\mathbb{Z} \) have? Explain your answer.

4. Denote the set of invertible elements of the ring \( \mathbb{Z}_n \) by \( U_n \).
   (a) List all the elements of \( U_{18} \).
   (b) Is \( U_{18} \) a cyclic group under multiplication? Justify your answer.

5. [NEW] Denote the set of invertible elements of the ring \( \mathbb{Z}_n \) by \( U_n \).
   (a) List all the elements of \( U_{24} \).
   (b) Is \( U_{24} \) a cyclic group under multiplication? Justify your answer.

6. [NEW] Find all positive integers \( n \) having the property that the group of units of \( \mathbb{Z}/n\mathbb{Z} \) is an elementary abelian 2-group.

7. Let \( U(R) \) denote the group of units of a ring \( R \). Prove that if \( m \) divides \( n \), then the natural ring homomorphism \( \mathbb{Z}_n \to \mathbb{Z}_m \) maps \( U(\mathbb{Z}_n) \) onto \( U(\mathbb{Z}_m) \).
   Give an example that shows that \( U(R) \) does not have to map onto \( U(S) \) under a surjective ring homomorphism \( R \to S \).
8. If $p$ is a prime satisfying $p \equiv 1 \pmod{4}$, then $p$ is a sum of two squares.

9. If $(\cdot)$ denotes the Legendre symbol, prove Euler’s Criterion: if $p$ is a prime and $a$ is any integer relatively prime to $p$, then $a^{(p-1)/2} \equiv \left( \frac{a}{p} \right) \pmod{p}$.

10. Let $R_1$ and $R_2$ be commutative rings with identities and let $R = R_1 \times R_2$. Show that every ideal $I$ of $R$ is of the form $I = I_1 \times I_2$ with $I_i$ an ideal of $R_i$ for $i = 1, 2$.

11. Show that a non-zero ring $R$ in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and is commutative.

12. Let $R$ be a finite commutative ring with more than one element and no zero-divisors. Show that $R$ is a field.

13. Determine for which integers $n$ the ring $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of fields. Prove your answer.

14. Let $R$ be a subring of a field $F$ such that for each $x$ in $F$ either $x \in R$ or $x^{-1} \in R$. Prove that if $I$ and $J$ are two ideals of $R$, then either $I \subseteq J$ or $J \subseteq I$.

15. The Jacobson Radical $J(R)$ of a ring $R$ is defined to be the intersection of all maximal ideals of $R$.
Let $R$ be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y$ in $R$.

16. Let $R$ be any ring with identity, and $n$ any positive integer. If $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in $R$, prove that $M_n(I)$ is an ideal of $M_n(R)$ whenever $I$ is an ideal of $R$, and that every ideal of $M_n(R)$ has this form.

17. Let $m$, $n$ be positive integers such that $m$ divides $n$. Then the natural map $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + (n) \mapsto a + (m)$ is a surjective ring homomorphism. If $U_n$, $U_m$ are the units of $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively, show that $\varphi : U_n \to U_m$ is a surjective group homomorphism.

18. Let $R$ be a ring with ideals $A$ and $B$. Let $R/A \times R/B$ be the ring with coordinate-wise addition and multiplication. Show the following.
(a) The map $R \to R/A \times R/B$ given by $r \mapsto (r + A, r + B)$ is a ring homomorphism.
(b) The homomorphism in part (a) is surjective if and only if $A + B = R$.

19. Let $m$ and $n$ be relatively prime integers.
(a) Show that if $c$ and $d$ are any integers, then there is an integer $x$ such that $x \equiv c \pmod{m}$ and $x \equiv d \pmod{n}$.
(b) Show that $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic as rings.

20. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$ such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

21. [NEW] Let $R$ be a commutative ring with identity and let $I_1, I_2, \ldots, I_n$ be pairwise co-maximal ideals of $R$ (i.e., $I_i + I_j = R$ if $i \neq j$). Show that $I_i + \bigcap_{j \neq i} I_j = R$ for all $i$.

22. Let $R$ be a commutative ring, not necessarily with identity, and assume there is some fixed positive integer $n$ such that $nx = 0$ for all $r \in R$. Prove that $R$ embeds in a ring $S$ with identity so that $R$ is an ideal of $S$ and $S/R \cong \mathbb{Z}/n\mathbb{Z}$. 

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23. Let \( R \) be a ring with identity 1 and \( a, b \in R \) such that \( ab = 1 \). Denote \( X = \{ x \in R \mid ax = 1 \} \).

Show the following.

(a) If \( x \in X \), then \( b + (1 - xa) \in X \).

(b) If \( \varphi : X \to X \) is the mapping given by \( \varphi(x) = b + (1 - xa) \), then \( \varphi \) is one-to-one.

(c) If \( X \) has more than one element, then \( X \) is an infinite set.

24. Let \( R \) be a commutative ring with identity and define \( U_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in R \right\} \).

Prove that every \( R \)-automorphism of \( U_2(R) \) is inner.

25. Let \( \mathbb{R} \) be the field of real numbers and let \( F \) be the set of all \( 2 \times 2 \) matrices of the form \( \begin{bmatrix} a & b \\ -3b & a \end{bmatrix} \), where \( a, b \in \mathbb{R} \). Show that \( F \) is a field under the usual matrix operations.

26. Let \( R \) be the ring of all \( 2 \times 2 \) matrices of the form \( \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \) where \( a \) and \( b \) are real numbers.

Prove that \( R \) is isomorphic to \( \mathbb{C} \), the field of complex numbers.

27. Let \( p \) be a prime and let \( R \) be the ring of all \( 2 \times 2 \) matrices of the form \( \begin{bmatrix} a & b \\ pb & a \end{bmatrix} \), where \( a, b \in \mathbb{Z} \). Prove that \( R \) is isomorphic to \( \mathbb{Z}[\sqrt{p}] \).

28. Let \( p \) be a prime and \( F_p \) the set of all \( 2 \times 2 \) matrices of the form \( \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), where \( a, b \in \mathbb{Z}_p \).

(a) Show that \( F_p \) is a commutative ring with identity.

(b) Show that \( F_7 \) is a field.

(c) Show that \( F_{13} \) is not a field.

29. Let \( I \subseteq J \) be right ideals of a ring \( R \) such that \( J/I \cong R \) as right \( R \)-modules. Prove that there exists a right ideal \( K \) such that \( I \cap K = (0) \) and \( I + K = J \).

30. A ring \( R \) is called simple if \( R^2 \neq 0 \) and 0 and \( R \) are its only ideals. Show that the center of a simple ring is 0 or a field.

31. Give an example of a field \( F \) and a one-to-one ring homomorphism \( \varphi : F \to F \) which is not onto. Verify your example.

32. Let \( D \) be an integral domain and let \( D[x_1, x_2, \ldots, x_n] \) be the polynomial ring over \( D \) in the \( n \) indeterminates \( x_1, x_2, \ldots, x_n \). Let

\[
V = \begin{bmatrix}
x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\
x_2^{n-1} & \cdots & x_2^2 & x_2 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
x_n^{n-1} & \cdots & x_n^2 & x_n & 1
\end{bmatrix}.
\]

Prove that the determinant of \( V \) is \( \prod_{1 \leq i < j \leq n} (x_i - x_j) \).
33. Let \( R = C[0,1] \) be the set of all continuous real-valued functions on \([0,1]\). Define addition and multiplication on \( R \) as follows. For \( f, g \in R \) and \( x \in [0,1] \),
\[
(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x)g(x).
\]

(a) Show that \( R \) with these operations is a commutative ring with identity.
(b) Find the units of \( R \).
(c) If \( f \in R \) and \( f^2 = f \), then \( f = 0_R \) or \( f = 1_R \).
(d) If \( n \) is a positive integer and \( f \in R \) is such that \( f^n = 0_R \), then \( f = 0_R \).

34. Let \( S \) be the ring of all bounded, continuous functions \( f : \mathbb{R} \to \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers. Let \( I \) be the set of functions \( f \) in \( S \) such that \( f(t) \to 0 \) as \( |t| \to \infty \).

(a) Show that \( I \) is an ideal of \( S \).
(b) Suppose \( x \in S \) is such that there is an \( i \in I \) with \( ix = x \). Show that \( x(t) = 0 \) for all sufficiently large \( |t| \).

35. Let \( \mathbb{Q} \) be the field of rational numbers and \( D = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \).

(a) Show that \( D \) is a subring of the field of real numbers.
(b) Show that \( D \) is a principal ideal domain.
(c) Show that \( \sqrt{3} \) is not an element of \( D \).

36. Show that if \( p \) is a prime such that \( p \equiv 1 \pmod{4} \), then \( x^2 + 1 \) is not irreducible in \( \mathbb{Z}_p[x] \).

37. Show that if \( p \) is a prime such that \( p \equiv 3 \pmod{4} \), then \( x^2 + 1 \) is irreducible in \( \mathbb{Z}_p[x] \).

38. Show that if \( p \) is a prime such that \( p \equiv 1 \pmod{6} \), then \( x^3 + 1 \) splits in \( \mathbb{Z}_p[x] \).

Prime, Maximal, and Primary Ideals

39. Let \( R \) be a non-zero commutative ring with 1. Show that an ideal \( M \) of \( R \) is maximal if and only if \( R/M \) is a field.

40. Let \( R \) be a commutative ring with 1. Show that an ideal \( P \) of \( R \) is prime if and only if \( R/P \) is an integral domain.

41. (a) Let \( R \) be a commutative ring with 1. Show that if \( M \) is a maximal ideal of \( R \) then \( M \) is a prime ideal of \( R \).
(b) Give an example of a non-zero prime ideal in a ring \( R \) that is not a maximal ideal.

42. Let \( R \) be a non-zero ring with identity. Show that every proper ideal of \( R \) is contained in a maximal ideal.

43. [NEW] Let \( M_1 \neq M_2 \) be two maximal ideals in the commutative ring \( R \) and let \( I = M_1 \cap M_2 \). Prove that \( R/I \) is isomorphic to the direct sum of two fields.

44. Let \( R \) be a non-zero commutative ring with 1. Show that if \( I \) is an ideal of \( R \) such that \( 1 + a \) is a unit in \( R \) for all \( a \in I \), then \( I \) is contained in every maximal ideal of \( R \).
45. **[NEW]** Let $R$ be a commutative ring with identity. Suppose $R$ contains an idempotent element $a$ other than 0 or 1. Show that every prime ideal in $R$ contains an idempotent element other than 0 or 1. (An element $a \in R$ is idempotent if $a^2 = a$.)

46. Let $R$ be a commutative ring with 1.
   (a) Prove that $(x)$ is a prime ideal in $R[x]$ if and only if $R$ is an integral domain.
   (b) Prove that $(x)$ is a maximal ideal in $R[x]$ if and only if $R$ is a field.

47. Find all values of $a$ in $\mathbb{Z}_3$ such that the quotient ring

\[
\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)
\]

is a field. Justify your answer.

48. Find all values of $a$ in $\mathbb{Z}_5$ such that the quotient ring

\[
\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)
\]

is a field. Justify your answer.

49. Let $R$ be a commutative ring with identity and let $U$ be maximal among non-finitely generated ideals of $R$. Prove $U$ is a prime ideal.

50. Let $R$ be a commutative ring with identity and let $U$ be maximal among non-principal ideals of $R$. Prove $U$ is a prime ideal.

51. Let $R$ be a non-zero commutative ring with 1 and $S$ a multiplicative subset of $R$ not containing 0. Show that if $P$ is maximal in the set of ideals of $R$ not intersecting $S$, then $P$ is a prime ideal.

52. Let $R$ be a non-zero commutative ring with 1.
   (a) Let $S$ be a multiplicative subset of $R$ not containing 0 and let $P$ be maximal in the set of ideals of $R$ not intersecting $S$. Show that $P$ is a prime ideal.
   (b) Show that the set of nilpotent elements of $R$ is the intersection of all prime ideals.

53. Let $R$ be a commutative ring with identity and let $x \in R$ be a non-nilpotent element. Prove that there exists a prime ideal $P$ of $R$ such that $x \not\in P$.

54. Let $R$ be a commutative ring with identity and let $S$ be the set of all elements of $R$ that are not zero-divisors. Show that there is a prime ideal $P$ such that $P \cap S$ is empty. (Hint: Use Zorn's Lemma.)

55. Let $R$ be a commutative ring with identity and let $C$ be a chain of prime ideals of $R$. Show that $\bigcup_{P \in C} P$ and $\bigcap_{P \in C} P$ are prime ideals of $R$.

56. Let $R$ be a commutative ring and $P$ a prime ideal of $R$. Show that there is a prime ideal $P_0 \subseteq P$ that does not properly contain any prime ideal.

57. Let $R$ be a commutative ring with 1 such that for every $x$ in $R$ there is an integer $n > 1$ (depending on $x$) such that $x^n = x$. Show that every prime ideal of $R$ is maximal.

58. Let $R$ be a commutative ring with 1 in which every ideal is a prime ideal. Prove that $R$ is a field. (Hint: For $a \neq 0$ consider the ideals $(a)$ and $(a^2)$.)
59. Let $D$ be a principal ideal domain. Prove that every nonzero prime ideal of $D$ is a maximal ideal.

60. Show that if $R$ is a finite commutative ring with identity then every prime ideal of $R$ is a maximal ideal.

61. Let $R = C[0, 1]$ be the ring of all continuous real-valued functions on $[0, 1]$, with addition and multiplication defined as follows. For $f, g \in R$ and $x \in [0, 1]$,

$$
(f + g)(x) = f(x) + g(x)
$$
$$
(fg)(x) = f(x)g(x).
$$

Prove that if $M$ is a maximal ideal of $R$, then there is a real number $x_0 \in [0, 1]$ such that $M = \{f \in R \mid f(x_0) = 0\}$.

62. Let $R$ be a commutative ring with identity, and let $P \subset Q$ be prime ideals of $R$. Prove that there exist prime ideals $P^*, Q^*$ satisfying $P \subseteq P^* \subset Q^* \subseteq Q$, such that there are no prime ideals strictly between $P^*$ and $Q^*$. HINT: Fix $x \in Q - P$ and show that there exists a prime ideal $P^*$ containing $P$, contained in $Q$ and maximal with respect to not containing $x$.

63. Let $R$ be a commutative ring with 1. An ideal $I$ of $R$ is called a primary ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geq 1$.

(a) Show that an ideal $I$ of $R$ is primary if and only if $R/I \neq 0$ and every zero-divisor in $R/I$ is nilpotent.

(b) Show that if $I$ is a primary ideal of $R$ then the radical $\text{Rad}(I)$ of $I$ is a prime ideal. (Recall that $\text{Rad}(I) = \{x \in R \mid x^n \in I$ for some $n\}$.)

Commutative Rings

64. Let $R$ be a commutative ring with identity. Show that $R$ is an integral domain if and only if $R$ is a subring of a field.

65. Let $R$ be a commutative ring with identity. Show that if $x$ and $y$ are nilpotent elements of $R$ then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in $R$.

66. Let $R$ be a commutative ring with identity. An ideal $I$ of $R$ is called irreducible if it cannot be expressed as the intersection of two ideals of $R$ neither of which is contained in the other. Show the following.

(a) If $P$ is a prime ideal then $P$ is irreducible.

(b) If $x$ is a non-zero element of $R$, then there is an ideal $I_x$, maximal with respect to the property that $x \notin I_x$, and $I_x$ is irreducible.

(c) If every irreducible ideal of $R$ is a prime ideal, then 0 is the only nilpotent element of $R$.

67. Let $R$ be a commutative ring with 1 and let $I$ be an ideal of $R$ satisfying $I^2 = \{0\}$. Show that if $a + I \in R/I$ is an idempotent element of $R/I$, then the coset $a + I$ contains an idempotent element of $R$.

68. Let $R$ be a commutative ring with identity that has exactly one prime ideal $P$. Prove the following.

(a) $R/P$ is a field.

(b) $R$ is isomorphic to $R_P$, the ring of quotients of $R$ with respect to the multiplicative set $R - P = \{s \in R \mid s \notin P\}$.
69. Let $R$ be a commutative ring with identity and $\sigma : R \to R$ a ring automorphism.
   (a) Show that $F = \{r \in R \mid \sigma(r) = r\}$ is a subring of $R$ and the identity of $R$ is in $F$.
   (b) Show that if $\sigma^2$ is the identity map on $R$, then each element of $R$ is the root of a monic polynomial of degree two in $F[x]$.

70. Let $R$ be a commutative ring with identity that has exactly three ideals, $\{0\}, I,$ and $R$.
   (a) Show that if $a \notin I$, then $a$ is a unit of $R$.
   (b) Show that if $a, b \in I$ then $ab = 0$.

71. Let $R$ be a commutative ring with 1. Show that if $u$ is a unit in $R$ and $n$ is nilpotent, then $u + n$ is a unit.

72. Let $R$ be a commutative ring with identity. Suppose that for every $a \in R$, either $a$ or $1 - a$ is invertible. Prove that $N = \{a \in R \mid a$ is not invertible$\}$ is an ideal of $R$.

73. Let $R$ be a commutative ring with 1. Show that the sum of any two principal ideals of $R$ is principal if and only if every finitely generated ideal of $R$ is principal.

74. Let $R$ be a commutative ring with identity such that not every ideal is a principal ideal.
   (a) Show that there is an ideal $I$ maximal with respect to the property that $I$ is not a principal ideal.
   (b) If $I$ is the ideal of part (a), show that $R/I$ is a principal ideal ring.

75. Recall that if $R \subseteq S$ is an inclusion of commutative rings (with the same identity) then an element $s \in S$ is integral over $R$ if $s$ satisfies some monic polynomial with coefficients in $R$. Prove the equivalence of the following statements.
   (i) $s$ is integral over $R$.
   (ii) $R[s]$ is finitely generated as an $R$-module.
   (iii) There exists a faithful $R[s]$ module which is finitely generated as an $R$-module.

76. Recall that if $R \subseteq S$ is an inclusion of commutative rings (with the same identity) then $S$ is an integral extension of $R$ if every element of $S$ satisfies some monic polynomial with coefficients in $R$. Prove that if $R \subseteq S \subseteq T$ are commutative rings with the same identity, then $S$ is integral over $R$ and $T$ is integral over $S$ if and only if $T$ is integral over $R$.

77. Let $R \subseteq S$ be commutative domains with the same identity, and assume that $S$ is an integral extension of $R$. Let $I$ be a nonzero ideal of $S$. Prove that $I \cap R$ is a nonzero ideal of $R$.

**Domains**

78. Suppose $R$ is a domain and $I$ and $J$ are ideals of $R$ such that $IJ$ is principal. Show that $I$ (and by symmetry $J$) is finitely generated.
   [Hint: If $IJ = (a)$, then $a = \sum_{i=1}^{n} x_i y_i$ for some $x_i \in I$ and $y_i \in J$. Show the $x_i$ generate $I$.]

79. [NEW] Prove that if $D$ is a Euclidean Domain, then $D$ is a Principal Ideal Domain.

80. Show that if $p$ is a prime such that there is an integer $b$ with $p = b^2 + 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.
81. Show that if \( p \) is a prime such that \( p \equiv 1 \pmod{4} \), then \( \mathbb{Z}[\sqrt{p}] \) is not a unique factorization domain.

82. Let \( D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\} \) — a subring of the field of real numbers and necessarily an integral domain (you need not show this) — and \( F = \mathbb{Q}(\sqrt{5}) \) its field of fractions. Show the following:
   (a) \( x^2 + x - 1 \) is irreducible in \( D[x] \) but not in \( F[x] \).
   (b) \( D \) is not a unique factorization domain.

83. Let \( D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\} \) and \( F = \mathbb{Q}(\sqrt{21}) \), the field of fractions of \( D \).
   Show the following:
   (a) \( x^2 - x - 5 \) is irreducible in \( D[x] \) but not in \( F[x] \).
   (b) \( D \) is not a unique factorization domain.

84. Let \( D = \mathbb{Z}(\sqrt{-11}) = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\} \) and \( F = \mathbb{Q}(\sqrt{-11}) \) its field of fractions. Show the following:
   (a) \( x^2 - x + 3 \) is irreducible in \( D[x] \) but not in \( F[x] \).
   (b) \( D \) is not a unique factorization domain.

85. Let \( D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\} \) and \( F = \mathbb{Q}(\sqrt{13}) \) its field of fractions. Show the following:
   (a) \( x^2 + 3x - 1 \) is irreducible in \( D[x] \) but not in \( F[x] \).
   (b) \( D \) is not a unique factorization domain.

86. Let \( D \) be an integral domain and \( F \) a subring of \( D \) that is a field. Show that if each element of \( D \) is algebraic over \( F \), then \( D \) is a field.

87. Let \( R \) be an integral domain containing the subfield \( F \) and assume that \( R \) is finite dimensional over \( F \) when viewed as a vector space over \( F \). Prove that \( R \) is a field.

88. Let \( D \) be an integral domain.
   (a) For \( a, b \in D \) define a greatest common divisor of \( a \) and \( b \).
   (b) For \( x \in D \) denote \( (x) = \{dx \mid d \in D\} \). Prove that if \( (a) + (b) = (d) \), then \( d \) is a greatest common divisor of \( a \) and \( b \).

89. Let \( D \) be a principal ideal domain.
   (a) For \( a, b \in D \), define a least common multiple of \( a \) and \( b \).
   (b) Show that \( d \in D \) is a least common multiple of \( a \) and \( b \) if and only if \( (a) \cap (b) = (d) \).

90. Let \( D \) be a principal ideal domain and let \( a, b \in D \).
   (a) Show that there is an element \( d \in D \) that satisfies the properties
      i. \( d|a \) and \( d|b \) and
      ii. if \( e|a \) and \( e|b \) then \( e|d \).
   (b) Show that there is an element \( m \in D \) that satisfies the properties
      i. \( a|m \) and \( b|m \) and
      ii. if \( a|e \) and \( b|e \) then \( m|e \).

91. Let \( R \) be a principal ideal domain. Show that if \( (a) \) is a nonzero ideal in \( R \), then there are only finitely many ideals in \( R \) containing \( (a) \).
92. Let $D$ be a unique factorization domain and $F$ its field of fractions. Prove that if $d$ is an irreducible element in $D$, then there is no $x \in F$ such that $x^2 = d$.

93. Let $D$ be a Euclidean domain. Prove that every non-zero prime ideal is a maximal ideal.

94. Let $\pi$ be an irreducible element of a principal ideal domain $R$. Prove that $\pi$ is a prime element (that is, $\pi | ab$ implies $\pi | a$ or $\pi | b$).

95. Let $\varphi : D \to \mathbb{N}$ be a Euclidean domain. Suppose $\varphi(a + b) \leq \max\{\varphi(a), \varphi(b)\}$ for all $a, b \in D$. Prove that $D$ is either a field or isomorphic to a polynomial ring over a field.

96. Let $D$ be an integral domain and $F$ its field of fractions. Show that if $g$ is an isomorphism of $D$ onto itself, then there is a unique isomorphism $h$ of $F$ onto $F$ such that $h(d) = g(d)$ for all $d$ in $D$.

97. Let $D$ be a unique factorization domain such that if $p$ and $q$ are irreducible elements of $D$, then $p$ and $q$ are associates. Show that if $A$ and $B$ are ideals of $D$, then either $A \subseteq B$ or $B \subseteq A$.

98. Let $D$ be a unique factorization domain and $p$ a fixed irreducible element of $D$ such that if $q$ is any irreducible element of $D$, then $q$ is an associate of $p$. Show the following.
   (a) If $d$ is a nonzero element of $D$, then $d$ is uniquely expressible in the form $up^n$, where $u$ is a unit of $D$ and $n$ is a non-negative integer.
   (b) $D$ is a Euclidean domain.

99. Prove that $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain.

100. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean ring and compute the greatest common divisor of $5 + i$ and $13$ using the Euclidean algorithm.

**Polynomial Rings**

101. Show that the polynomial $f(x) = x^4 + 5x^2 + 3x + 2$ is irreducible over the field of rational numbers.

102. Let $D$ be an integral domain and $D[x]$ the polynomial ring over $D$. Suppose $\varphi : D[x] \to D[x]$ is an isomorphism such that $\varphi(d) = d$ for all $d \in D$. Show that $\varphi(x) = ax + b$ for some $a, b \in D$ and that $a$ is a unit of $D$.

103. Let $f(x) = a_0 + a_1x + \cdots + a_kx^k \in \mathbb{Z}[x]$ and $p$ a prime such that $p | a_i$ for $i = 1, \ldots, k - 1$, $p \nmid a_k$, $p \nmid a_n$, and $p^2 \nmid a_0$. Show that $f(x)$ has an irreducible factor in $\mathbb{Z}[x]$ of degree at least $k$.

104. Let $D$ be an integral domain and $D[x]$ the polynomial ring over $D$ in the indeterminate $x$. Show that if every nonzero prime ideal of $D[x]$ is a maximal ideal, then $D$ is a field.

105. Let $R$ be a commutative ring with 1 and let $f(x) \in R[x]$ be nilpotent. Show that the coefficients of $f$ are nilpotent.

106. Show that if $R$ is an integral domain and $f(x)$ is a unit in the polynomial ring $R[x]$, then $f(x)$ is in $R$. 

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107. Let $D$ be a unique factorization domain and $F$ its field of fractions. Prove that if $f(x)$ is a monic polynomial in $D[x]$ and $\alpha \in F$ is a root of $f$, then $\alpha \in D$.

108. (a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.

(b) Show that $x^4 + 1$ is not irreducible in $\mathbb{Z}_3[x]$.

109. Let $F[x, y]$ be the polynomial ring over a field $F$ in two indeterminates $x$, $y$. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

110. Let $F$ be a field. Prove that the polynomial ring $F[x]$ is a PID and that $F[x, y]$ is not a PID.

111. Let $D$ be an integral domain and let $c$ be an irreducible element in $D$. Show that the ideal $(x, c)$ generated by $x$ and $c$ in the polynomial ring $D[x]$ is not a principal ideal.

112. [CORRECTED] Show that if $R$ is a commutative ring with 1 that is not a field, then $R[x]$ is not a principal ideal domain.

113. (a) Let $\mathbb{Z}[\frac{1}{2}] = \{ \frac{a}{2^n} \mid a, n \in \mathbb{Z}, n \geq 0 \}$, the smallest subring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\frac{1}{2}$.

Let $(2x - 1)$ be the ideal of $\mathbb{Z}[x]$ generated by the polynomial $2x - 1$.

Show that $\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}[\frac{1}{2}]$.

(b) Find an ideal $I$ of $\mathbb{Z}[x]$ such that $(2x - 1) \subsetneq I \subsetneq \mathbb{Z}[x]$.

Non-commutative Rings

114. Let $R$ be a ring with identity such that the identity map is the only ring automorphism of $R$. Prove that the set $N$ of all nilpotent elements of $R$ is an ideal of $R$.

115. Let $p$ be a prime. A ring $S$ is called a $p$-ring if the characteristic of $S$ is a power of $p$. Show that if $R$ is a ring with identity of finite characteristic, then $R$ is isomorphic to a finite direct product of $p$-rings for distinct primes.

116. Let $R$ be a ring.

(a) Show that there is a unique smallest (with respect to inclusion) ideal $A$ such that $R/A$ is a commutative ring.

(b) Give an example of a ring $R$ such that for every proper ideal $I$, $R/I$ is not commutative. Verify your example.

(c) For the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ with the usual matrix operations, find the ideal $A$ of part (a).

117. If $R$ is any ring with identity, let $J(R)$ denote the Jacobson radical of $R$. Show that if $e$ is any idempotent of $R$, then $J(eRe) = eJ(R)e$.

118. If $n$ is a positive integer and $F$ is any field, let $M_n(F)$ denote the ring of $n \times n$ matrices with entries in $F$. Prove that $M_n(F)$ is a simple ring. Equivalently, $\text{End}_F(V)$ is a simple ring if $V$ is a finite dimensional vector space over $F$. 

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119. A ring $R$ is nilpotent-free if $a^n = 0$ for $a \in R$ and some positive integer $n$ implies $a = 0$.

(a) Suppose there is an ideal $I$ such that $R/I$ is nilpotent-free. Show there is a unique smallest (with respect to inclusion) ideal $A$ such that $R/A$ is nilpotent-free.

(b) Give an example of a ring $R$ such that for every proper ideal $I$, $R/I$ is not nilpotent-free. Verify your example.

(c) Show that if $R$ is a commutative ring with identity, then there is a proper ideal $I$ of $R$ such that $R/I$ is nilpotent-free, and find the ideal $A$ of part (a).

**Local Rings, Localization, Rings of Fractions**

120. Let $R$ be an integral domain. Construct the field of fractions $F$ of $R$ by defining the set $F$ and the two binary operations, and show that the two operations are well-defined. Show that $F$ has a multiplicative identity element and that every nonzero element of $F$ has a multiplicative inverse.

121. A local ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring $R$ is local if and only if the set of non-units in $R$ is an ideal.

122. Let $R$ be a commutative ring with 1 $\neq 0$ in which the set of nonunits is closed under addition. Prove that $R$ is local, i.e., has a unique maximal ideal.

123. Let $D$ be an integral domain and $F$ its field of fractions. Let $P$ be a prime ideal in $D$ and $D_P = \{ab^{-1} \mid a, b \in D, b \notin P\} \subseteq F$. Show that $D_P$ has a unique maximal ideal.

124. Let $R$ be a commutative ring with identity and $P$ a prime ideal of $R$. Let $R_P$ be the ring of quotients of $R$ with respect to the set $R - P = \{s \in R \mid s \notin P\}$. Show that $R_P$ has a unique maximal ideal.

125. Let $R$ be an integral domain, $S$ a multiplicative set, and let $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ (contained in the field of fractions of $R$). Show that if $P$ is a prime ideal of $R$, then $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

126. Let $R$ be a commutative ring with identity and $P$ a prime ideal of $R$. Let $R_P$ be the ring of quotients of $R$ with respect to the set $R - P = \{s \in R \mid s \notin P\}$. Show that $R_P/P_P$ is the field of fractions of the integral domain $R/P$.

127. Let $D$ be an integral domain and $F$ its field of fractions. Denote by $\mathcal{M}$ the set of all maximal ideals of $D$. For $M \in \mathcal{M}$, let $D_M = \{\frac{a}{s} \mid a, s \in D, s \notin M\} \subset F$. Show that $\bigcap_{M \in \mathcal{M}} D_M = D$.

128. Let $R$ be a commutative ring with 1 and $D$ a multiplicative subset of $R$ containing 1. Let $J$ be an ideal in the ring of fractions $D^{-1}R$ and let

$$I = \{a \in R \mid \frac{a}{d} \in J \text{ for some } d \in D\}.$$ 

Show that $I$ is an ideal of $R$.

129. Let $D$ be a principal ideal domain and let $P$ be a non-zero prime ideal. Show that $D_P$, the localization of $D$ at $P$, is a principal ideal domain and has a unique irreducible element, up to associates.
Chains and Chain Conditions

130. Let $R$ be a commutative ring with identity. Prove that any non-empty set of prime ideals of $R$ contains maximal and minimal elements.

131. Let $R$ be a commutative ring with 1. We say $R$ satisfies the ascending chain condition if whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an ascending chain of ideals, there is an integer $N$ such that $I_k = I_N$ for all $k \geq N$. Show that $R$ satisfies the ascending chain condition if and only if every ideal of $R$ is finitely generated.

132. [NEW] Define Noetherian ring and prove that if $R$ is Noetherian, then $R[x]$ is Noetherian.

133. Let $R$ be a commutative Noetherian ring with identity. Prove that there are only finitely many minimal prime ideals of $R$.

134. [NEW] Let $R$ be a commutative Noetherian ring in which every 2-generated ideal is principal. Prove that $R$ is a Principal Ideal Domain.

135. Let $R$ be a commutative Noetherian ring with identity and let $I$ be an ideal in $R$. Let $J = \text{Rad}(I)$. Prove that there exists a positive integer $n$ such that $j^n \in I$ for all $j \in J$.

136. Let $R$ be a commutative Noetherian domain with identity. Prove that every nonzero ideal of $R$ contains a product of nonzero prime ideals of $R$.

137. Let $R$ be a ring satisfying the descending chain condition on right ideals. If $J(R)$ denotes the Jacobson radical of $R$, prove that $J(R)$ is nilpotent.

138. Show that if $R$ is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

139. Let $P$ be a nonzero prime ideal of the commutative Noetherian domain $R$. Assume $P$ is principal. Prove that there does not exist a prime ideal $Q$ satisfying $(0) < Q < P$.

140. Let $R$ be a commutative Noetherian ring. Prove that every nonzero ideal $A$ of $R$ contains a product of prime ideals (not necessarily distinct) each of which contains $A$.

141. Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module that is not Artinian (Noetherian, of finite composition length). Let $\mathcal{I}$ be the set of ideals $I$ of $R$ such that there exists an $R$-submodule $N$ of $M$ with the property that $N/NI$ is not Artinian (Noetherian, of finite composition length, respectively). Show that if $A \in \mathcal{I}$ is a maximal element of $\mathcal{I}$, then $A$ is a prime ideal of $R$. 