QUALIFYING EXAM IN ALGEBRA

August 2000

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. A matrix A has characteristic polynomial $\Delta(x) = (x-3)^5$ and minimal polynomial $m(x) = (x-3)^3$.

   (a) List all possible Jordan canonical forms for A.

   (b) Determine the Jordan canonical form of the matrix

   $$A = \begin{bmatrix} 3 & -1 & 2 & 0 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 2 & -3 & 0 & 3 \end{bmatrix}$$

   which has the given characteristic and minimal polynomials.

2. Let V be a finite dimensional vector space and $T : V \to V$ a *non-zero* linear operator. Show that if $\ker T = \operatorname{Im} T$, then $\dim V$ is an *even* integer and the minimal polynomial of T is $m(x) = x^2$.

3. Let V be a finite-dimensional vector space over a field F and let U be a subspace. Show that there is a subspace W of V such that $V = U \oplus W$.

II. Group Theory

1. Let G be a finite group, p a prime, and P a Sylow p-subgroup of G. Let H be a subgroup of G that contains the normalizer $N_G(P)$ of P in G. Show that if g is an element of G such that $g^{-1}Pg \leq H$, then g is an element of H.

2. Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Show that if the order of H is relatively prime to the index of N in G, then $H \subseteq N$.

3. Show that if G is a simple group of order greater than 60, then G has no proper subgroup of index less than or equal to 5.

4. Show that a group of order $3^3 \cdot 5 \cdot 13$ must have a normal Sylow 13-subgroup or a normal Sylow 5-subgroup. [Hint: Show that if a Sylow 13-subgroup is not normal, then a Sylow 13-subgroup must normalize a Sylow 5-subgroup. Consider the normalizer of a Sylow 5-subgroup.]

5. (a) Prove that the additive group of the rational numbers is not cyclic.

   (b) Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

III. Ring Theory

1. Let p be a prime and let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ pb & a \end{bmatrix}$, where $a, b \in \boldsymbol{Z}$. Prove that R is isomorphic to $\boldsymbol{Z}[\sqrt{p}]$.

2. Let R be a commutative ring with identity. Suppose that for every $a \in R$, either a or $1 - a$ is invertible. Prove that $N = \{a \in R \,|\, a \text{ is not invertible}\}$ is an ideal of R.

3. Show that if R is a finite commutative ring with identity then every prime ideal of R is a maximal ideal.

4. Let D be a unique factorization domain such that if p and q are irreducible elements of D, then p and q are associates. Show that if A and B are ideals of D, then either $A \subseteq B$ or $B \subseteq A$.

5. Let D be an integral domain and F a subring of D which is a field. Show that if each element of D is algebraic over F, then D is a field.

IV. Field Theory

1. Let K be a field extension of F of degree n and let $f(x) \in F[x]$ be an irreducible polynomial of degree $m > 1$. Show that if m is relatively prime to n, then f has no root in K.

2. A field F is called *perfect* if every element of an algebraic closure of F is separable over F.

   Let F be a field of characteristic p. Show that the following are equivalent.

   (i) The field F is perfect.

   (ii) The map $a \mapsto a^p$ is an automorphism of F.

3. Let $u = \sqrt{2+\sqrt{2}}$, $v = \sqrt{2-\sqrt{2}}$, and $E = \boldsymbol{Q}(u)$, where $\boldsymbol{Q}$ is the field of rational numbers.

   (a) Find the minimal polynomial $f(x)$ of u over $\boldsymbol{Q}$.

   (b) Show $v \in E$. Hence conclude that E is a splitting field of $f(x)$ over $\boldsymbol{Q}$.

   (c) Show that the Galois group of E over $\boldsymbol{Q}$ is cyclic of order 4.

4. Let K be a Galois extension of k and let $k \subseteq F \subseteq K$ and $k \subseteq L \subseteq K$.

   (a) Show that $\mathrm{Gal}(K/LF) = \mathrm{Gal}(K/L) \cap \mathrm{Gal}(K/F)$.

   (b) Show that $\mathrm{Gal}(K/L \cap F) = \langle \mathrm{Gal}(K/L), \mathrm{Gal}(K/F) \rangle$.

5. Let F be a finite field. Show that the product of all the non-zero elements of F is -1.