QUALIFYING EXAM IN ALGEBRA

August 2002

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

    I. Linear Algebra — 1 problem
    II. Group Theory — 3 problems
    III. Ring Theory — 2 problems
    IV. Field Theory — 3 problems
    Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let $A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$

   (a) Determine conditions on a, b, c, and d so that there is only one Jordan block for each eigenvalue of A in the Jordan canonical form of A.

   (b) Suppose now $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

2. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\mathrm{im}\ T) = \dim(V)$.

3. Let V be a finite dimensional vector space over the field F. Let V^* be the dual space of V (that is, V^* is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$.

II. Group Theory

1. Let G be a finite group, H be a subgroup of G, and P be a Sylow p-subgroup of H for some prime p. Show that if H contains the normalizer $N_G(P)$ of P, then P is a Sylow p-subgroup of G.

2. Let N_1, N_2, and N_3 be normal subgroups of a group G and assume that for $i \neq j$, $N_i \cap N_j = \langle 1 \rangle$ and $N_i N_j = G$. Show that G is isomorphic to $N_1 \times N_1$ and G is abelian.

3. Let G be a finite group. Prove that G is a cyclic p-group, for some prime p, if and only if G has exactly one conjugacy class of maximal subgroups.

4. Let G be a finite simple group containing an element of order 21. Show that every proper subgroup of G has index at least 10.

5. Show that if G is a group of order $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, then G has an abelian subgroup of index 2.

III. Ring Theory

1. Let F be any field. Show that the ring $M_2(F)$ of all 2×2 matrices with entries in F is a simple ring.

2. Show that if R is a finite commutative ring with identity then every prime ideal of R is a maximal ideal.

3. Show that if p is a prime such that $p \equiv 1 (\mathrm{mod}\ 4)$, then $\boldsymbol{Z}[\sqrt{p}]$ is not a unique factorization domain.

4. Let $F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

5. Let R be a commutative ring with identity and let $\mathcal{C}$ be a chain of prime ideals of R. Show that $\bigcup_{C \in \mathcal{C}} C$ and $\bigcap_{C \in \mathcal{C}} C$ are prime ideals of R.

IV. Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{5 + \sqrt{3}}$ over the field $\boldsymbol{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

2. Let F be a field, let $E = F(a)$ be a simple extension field of F, and let $b \in E - F$. Prove that a is algebraic over $F(b)$.

3. Let F be a field and E a splitting field of the irreducible polynomial $f(x) \in F[x]$. Show that if $c, d \in F$ and $c \neq 0$, then the polynomial $f(cx + d)$ splits in $E[x]$.

4. Show that every finite field is perfect. (Recall that a field F of characteristic p is called *perfect* if the map $\alpha \mapsto \alpha^p$ is a surjection on F.)

5. Let $\boldsymbol{Q}$ be the field of rational numbers and η a complex primitive 8th root of unity. Determine $\mathrm{Gal}(\boldsymbol{Q}(\eta)/\boldsymbol{Q})$ and all the intermediate fields between $\boldsymbol{Q}$ and $\boldsymbol{Q}(\eta)$.